Everymind's
Algebraic Artifice Vindicated

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Dedication

This book is dedicated to

Isaac Todhunter

who wrote perhaps the
best mathematics texts for self-study
and who certainly wrote the
best answer keys.

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Preface

Think of this book as a mini-master-class in fundamental algebra transformations. Fundamental or basic algebra once filled large college textbooks like Todhunter's *Algebra for the Use of Colleges and Schools*, which is used in this text, or Chrystal's two-volume double doorstop of *Algebra - An Elementary Text Book*. Both of these contain more algebra than one encounters in all of public school and a bachelor's degree in Mathematics from a respected university with a respected math department.

In this master-class, Isaac Todhunter, F.R.S, of Cambridge University is the master. I am merely *his* amanuensis and *your* peanut gallery. The book itself is a series of problems with solutions where Todhunter shows that the form of number in the problem has a multiplicity and that by handling the transformation using another of its forms or by conforming it to an alternative basic form one can more easily transform the problem at hand.

I seem to have an undying interest in basic and fundamental mathematics and even more interest in the form of number. The first explains why I am still fiddling around with this kind of algebra. The second explains my contribution to this text -- which will be comments on Todhunter's solutions as they bear on my idea of the form of number. You will see what I mean by that as you read the comments. Or you may safely ignore the comments and just enjoy Todhunter's skill and understanding. His understanding is the reason this book exists.

As for our title, Todhunter calls these methods of solution "algebraic artifice." Lesser men and women, usually teachers, call such things "tricks." They are not "tricks" nor are they "artifice". They are the product of a deep understanding of algebra and a realization of the multiplicity of forms. They are the expressions of a mature understanding of mathematics.

I will assume that the reader's knowledge of algebra is equivalent to what is covered in *Everymind's De Morgan's Elements* and keep everything within that reach.

Introduction

This is a completely unoriginal book. Let me explain.

Not that long ago, as time passes for a hexagenarian, I worked through Todhunter's college algebra text, every page and every problem. Actually, at his suggestion, it was every third problem. But every one of those. Todhunter's books were written to be used as self-study texts and often had twin answer key texts. And Todhunter, as I mentioned in the dedication, writes the best answer keys.

If the problem is straightforward, he provides no answer. If you can't get that far, he seems to say, you have taken the wrong train. Go study Latin. If the problem has one or more difficulties, he begins the answer with some line in the middle of the solution, which he expects you to arrive at, and then walks you through the answer from there until you reach the point where anyone should be able to go on and complete the solution. But what I really enjoyed was his third approach. He would say, in so many words, that you could solve this in the ordinary way *or thus: ... .* And his *or thus* would reveal that the problem could be viewed as falling under another and, perhaps, unexpected form of number which could be exploited to your advantage. I really enjoyed these kinds of answers even though they were rare. Out of 1100 problems, less than 30 are treated in this way.

I had thought of pulling all of these third type of solutions out of his answer key and including them in a text on basic algebra. But in creating *Everymind's De Morgan's Elements* , I found I had covered almost all of **that** algebra and didn't want to dilute De Morgan with Todhunter. And looking ahead at what I might be doing in future writings, I saw no place for these sweet revelations of Todhunter. When I write a mathematics text, a primary motive is to write it as a means of solidifying these mathematics in my own understanding. My other primary motive, as a longtime GNU-head, is to share what I've learned with everyone else who's interested. And I still wanted to go through Todhunter's answer key and nail down what he had revealed in these solutions of alternative forms.

So I decided to do this short text of his problems and answers and of my notes-to-self, disguised as public commentary, to nail down what I could gain from Todhunter's understanding before I went on to some longer Everymind's text. Selfishly, this book is for my own study of the fundaments and forms. Unselfishly, it is for everyone else who enjoys algebra. This "older" approach to algebra of Todhunter's has been supplanted by abstract algebra in many ways, resulting in this more fundamental form being compressed into almost nothing in the schools. So this text may have something even for those with advanced degrees, who might possibly be seeing some of this for the first time. I think every mathematician or student of mathematics should be able to find goodness in Todhunter's generous work. This text is simply one problem and solution after another copied from his text. All of the original work here is Todhunter's. All of the (possibly annoying) comments are mine. Throughout, I maintain his chapter titles and problem numbers in case anyone wishes to investigate further.

Fractions

48. Multiply $x^2 - x + 1$ by $1/x^2 + 1/x + 1$.

Solution:

Multiply $x^2 - x + 1$ by $1/x^2 + 1/x + 1$ by ordinary work. Or thus:

$(x^2 - x + 1) \times (1/x^2 + 1/x + 1)$

$= (x^2 - x + 1) \times (1 + x + x^2)/x^2$

$= \dfrac{(x^2 + 1)^2 - x^2}{x^2}$ = &c.

* * *

Traditional algebra texts would have lists of "important formulae" beginning with

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

and leading up to things like

$$(a + b - c)(b + c - a)(c + a - b) = \text{[Exercise for the reader.]}$$

and mathematicians memorized these formulae. Or rather, forms. Of number. The form in this problem is

$$(a + b + c)(a - b + c) = a^2 - b^2 + c^2$$

Many of these memorized forms were symmetric and easy to remember. So Todhunter sees this form, latent in the problem, and brings it out.

58. Divide $x^2 + 1/x^2 + 2$ by $x + 1/x$.

Solution:

Divide $x^2 + 1/x^2 + 2$ by $x + 1/x$ by ordinary work. Or thus:

$$x^2 + 2 + 1/x^2 = \frac{x^4 + 2x^2 + 1}{x^2} = \frac{(x^2 + 1)^2}{x^2}$$

$$x + 1/x = \frac{x^2 + 1}{x}$$

* * *

I think two things are going on here:

Todhunter habitually brings this kind of expression into a fraction in his mind's eye. He did it in the last problem. Rather than let the complexity of terms muddle his thought, he unifies it. And he orders it normally in powers of x.

And then he considers the resulting form, which here is

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

Of course, you can only consider the resulting form to the extent that your mind is full of forms. You can't see $(a + b - c)(b + c - a)(c + a - b)$ in the world if it isn't already in your consciousness.

Quadratics

38. Solve as a quadratic: $\dfrac{x}{x-1} = \dfrac{3}{2} + \dfrac{x-1}{x}$

Solution:

$2x^2 = 3x(x - 1) + 2(x - 1)^2$, &c.
Or we may proceed thus:
Put y for x/(x - 1), then the equation becomes y = 3/2 + 1/y;
therefore $y^2 = 3/2 \cdot y + 1$.
By solving this quadratic in the usual way we obtain y = 2 or -1/2.
Taking the former value we have x/(x-1) = 2, which give x = 2;
taking the latter value we have x/(x-1) = -1/2, which gives x = 1/3.

* * *

Todhunter points out six other problems following this one which could be solved in this way, saying of this substitution, *it will in some cases diminish the work.* These examples all have terms in the form of a/b and b/a where a,b are monomials or binomials like

$$\frac{2x-3}{3x-5} \quad \text{and} \quad \frac{3x-5}{2x-3}$$

where the point is that you can sub in y and 1/y and then multiply by y to get a quadratic. Changing the form of the variable is always a choice. And in *De Morgan's Elements*, we saw that the choices were made in order to arrive at certain ends. Here the end is simply the basic quadratic form. But the form must be understood in order to inform the choice of form given to the variable. A couple of substitution examples from the next chapter:

24. $2^{x+1} + 4^x = 80$

Soln: $2^{x+1} + 2^{2x} = 80$; put y for 2^x, thus $2y + y^2 = 80$; solve this quadratic in y.

35. $\dfrac{x^2 - a^2}{x^2 + a^2} + \dfrac{x^2 + a^2}{x^2 - a^2} = \dfrac{34}{15}$ and here we have the y + 1/y we just spoke of.

75. $(x + a)(x + 2a)(x + 3a)(x + 4a) = c^4$

Soln: That is $(x^2 + 5ax + 2a^2)(x^2 + 5ax + 6a^2) = c^4$. Put y for $x^2 + 5ax$;
therefore $(y + 4a^2)(y + 6a^2) = c^4$; therefore $y^2 + 10a^2y + 25a^4 = c^4 + a^4$, &c.

48. Solve as a quadratic: $\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}$

Solution:

(x + 3)(x - 2)(x - 1) + (x - 3)(x + 2)(x - 1) = (x + 2)(x - 2)(2x - 3), &c. Or thus:

$$\frac{x+3}{x+2} = \frac{x+2+1}{x+2} = 1 + \frac{1}{x+2}.$$

and treating the other fractions similarly we have

$$1 + \frac{1}{x+2} + 1 - \frac{1}{x-2} = 2 - \frac{1}{x-1}\text{; therefore } \frac{1}{x+2} + \frac{1}{x-2} = \frac{1}{x-1}\text{, \&c.}$$

* * *

... which saves us from solving third degree equations which, in Todhunter's text, we aren't ready for. And we should see the form of a third degree equation in the problem itself and immediately look for a way of reducing it to the form of a quadratic.

Of course, textbook problems are constructed puzzles and the third degree terms would cancel if we went that route. But **looking for** the desired form of number leads to more progress than a willingness to do everything by brute force.

61. Solve as a quadratic: $\dfrac{x+a}{x-a} + \dfrac{x+b}{x-b} + \dfrac{x+c}{x-c} = 3$

Solution:

Clear of fractions and simplify; then

$$x^2(a + b + c) - 2x(bc + ca + ab) + 3abc = 0, \text{ \&c.}$$

Or the process may with advantage be conducted thus:

$$\frac{x+a}{x-a} - 1 + \frac{x+b}{x-b} - 1 + \frac{x+c}{x-c} - 1 = 0 \text{ ; therefore}$$

$$\frac{2a}{x-a} + \frac{2b}{x-b} + \frac{2c}{x-c} = 0 \qquad [\, 1 = (x - a)/(x - a) \text{ \&c.} \,]$$

therefore $a(x - b)(x - c) + b(x - c)(x - a) + c(x - a)(x - b) = 0$

* * *

The only form I see here is the form of algebraic fractions and a quick simplification of them. The form that interests me here is the first line where the literal coefficients (coeffs) take the form of the actual roots in their relation to the quadratic and are then multiplied by 1, -2, and 3. Familiarity with this form and with how the additional coeffs modify it might lead to an even easier solution.

Equations Reducible to Quadratics

38. $\mathbf{x^2 + 1/x^2 - a^2 - 1/a^2 = 0}$

Solution:

$x^4 - x^2(a^2 + 1/a^2) = -1$; therefore

$x^4 - x^2(a^2 + 1/a^2) + \frac{1}{4}(a^2 + 1/a^2)^2 = -1 + \frac{1}{4}(a^2 + 1/a^2)^2 = \frac{1}{4}(a^2 - 1/a^2)^2$, &c.

Or thus: $x^2 - a^2 = 1/a^2 - 1/x^2 = \frac{x^2 - a^2}{a^2x^2}$; $(x^2 - a^2)(1 - 1/a^2x^2) = 0$, &c.

* * *

What Todhunter recognizes here is the possibility of pulling the $(x^2- a^2)$ out as a factor with roots $\pm a$. Then he need only to solve $1 - 1/a^2x^2 = 0$ for remaining roots. It is the familiarity with form, both through practice and within memory, that allows us to see these possibilities.

Arithmetical Progression

36. The sum of m terms of an A.P. is n, and the sum of n terms with the same first term and the same common difference is m. Show that the sum of m+n terms is -(m+n) and the sum of m-n terms is (m - n)(1 + 2n/m)

Soln:

$n = m/2\cdot(2a + (m-1)b)$ [1]
$m = n/2\cdot(2a + (n-1)b)$ [2]
From these equations we may deduce
$b = -(2(m+n))/mn$ $a = (m^2 + n^2 + mn - m - n)/mn$
and then the sum of any assigned number of terms may be found.
Or we may proceed thus:
The sum of m + n terms = $(m+n)/2\cdot(2a + (m + n - 1)b$.
Now from [1] and [2] by subtraction
$2(n - m) = 2a(m - n) + (m - n)(m + n - 1)b$
Divide by m - n; thus $(m + n - 1)b + 2a = -2$.
Hence the sum of m + n terms = $-(m + n)$
Again, the sum of m - n terms = $(m - n)/2\cdot(2a +(m - n - 1)b$
Now we have just seen that $-1 = a + ((m+n-1)b)/2$,
and from [1] $2n/m = 2a + (m-1)b$;
Therefore by subtraction we obtain $2n/m + 1 = a + ((m-1-n)b)/2$
Hence the sum of m - n terms = $(m-n)(1 + 2n/m)$

* * *

The first method leads to calculation and perhaps generalization of results. The second method allows comparison and transformation towards the desired forms in order to reach the conclusions.

42. Sum to n terms 1 - 3 + 5 - 7 + ⋯

Solution:

Here we might first sum the series 1 + 5 + 9 + ⋯, and then the series 3 + 7 + 11 + ⋯; and subtract the second result from the first. Or we may proceed thus:

The first and second terms together make -2, the third and fourth terms together make -2 and so on. Thus if n be even the sum is $-2 \times n/2$, that is -n; and if n be odd the sum is -(n-1)+the last term, that is -(n - 1) + 2n - 1, that is n. Thus the sum is -n if n be even, and +n if n be odd; that is the sum is $-n(-1)^n$.

* * *

As with De Morgan (Todhunter's mentor in many ways), Todhunter examines the series to see what he can do with short sums of terms. De Morgan used this as an easy test of divergence -- if every so many terms summed up to at least some finite number then the series must diverge. This is investigating the actual form of the series rather than puzzling over it at face value or attacking it with brute force.

51. Divide unity into four parts in A.P. of which the sum of the cubes shall be 1/10.

Solution:

Let the four parts be denoted by $x - 3y, x - y, x + y, x + 3y$.
These are in A.P., the common difference being $2y$.
Thus as their sum is unity $4x = 1$
Also $(x - 3y)^3 + (x - y)^3 + (x + y)^3 + (x + 3y)^3 = 1/10$
This reduces to $4x^3 + 60xy^2 = 1/10$
Substitute the value of x and we obtain $y^2 = 1/400 \therefore y = \pm 1/20$.

* * *

I'll let Todhunter explain:

It will be seen that owing to the peculiar form we have adopted for the unknown quantities the work is much simplified. The artifice should be noticed as it is often useful. If we are engaged for example on a problem respecting three unknown quantities in A.P. instead of denoting them by x, $x + y$, $x + 2y$ it may be advantageous to denote them by $x - y$, x, $x + y$.

Underlying this in part is the form of $(a + b)(a - b) = a^2 - b^2$.

Binomial Theorem. Positive Integral Exponent

14. Show that the difference between the coefficients of x^{r+1} and x^r in the expansion of $(1 + x)^{n+1}$ is equal to the difference between the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1 + x)^n$.

Solution:

This result may be obtained by direct work. Or we may proceed thus:
$(1 + x)^{n+1}(1 - x) = (1 + x)^n(1 - x^2)$
Hence we infer that the coefficient of any assigned power of x will be the same whether it is obtained from the left-hand expression of from the right-hand expression. Now in the expansion of $(1 + x)^{n+1}(1 - x)$ the coefficient of x^{r+1} will be found by subtracting the coefficient of x^r in the expansion of $(1 + x)^{n+1}$ from the coefficient of x^{r+1}, &c.

* * *

This shows Todhunter's mastery over the form of the binomial expansion. And the "&c." at the end shows that he considers this understanding within the capacity of any high school or college student.

10. (p.492) If x be any prime number, except 2, the integral part of $(1 + \sqrt{2})^x$, diminished by 2, is divisible by 4x.

Solution:

Let I denote the integral part of $(1 + \sqrt{2})^x$ and F the fractional part.
By the Binomial Theorem:

$$I + F = 1 + x \cdot 2^{1/2} + x(x-1)/2 \cdot 2 + x(x-1)(x-2)/3! \cdot 2^{3/2} + \cdots$$

Now $1 - \sqrt{2}$ is a negative proper fraction,
therefore $(1 - \sqrt{2})^x$ is a negative proper fraction. Denote it by -F'; thus

$$-F' = 1 - x \cdot 2^{1/2} + x(x-1)/2 \cdot 2 - x(x-1)(x-2)/3! \cdot 2^{3/2} + \cdots$$

By addition, $I + F - F' = 2 + 2 \cdot x(x-1)/2 \cdot 2 + \cdots$
Hence F - F' must be zero and I - 2 is equal to a series of terms every one of which is divisible by $2x \cdot 2$, that is by 4x.

* * *

This idea of I + F is a useful exploitation of form. If you have a + b and a > b, then for some fraction F, b = Fa. And a + b becomes a + Fa. And if you have to expand $(a + Fa)^n$, then you can factor out the a and have $a^n(1 + F)^n$. Other uses will occur to you.

Multinomial Theorem

16. Find the coefficient of x^4 in the multinomial expansion of
$(1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots)$

Solution:

It is easy to see that $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$
$= (1 + x)(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots) = (1 + x)(1 - x)^{-2}$
Hence we require the coefficient of x^4 in $(1 + x)^7(1 - x)^{-14}$;
and we have only to expand the two factors by the Binomial Theorem
and multiply the results together.
Or proceeding as usual we have [a messier computation with lots of combinatorics]

* * *

By "easy to see" we understand a familiarity with the form of $1/(1-x)^n$ in its serial form. From De Morgan, we know $1 + x + x^2 + \cdots$ with $x\in(0,1)$ gives $1/(1-x)$. But Todhunter obviously sees $n = -2$ in $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$ and the factor of $1 + x$ in $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$. And above all, he finds it natural to simply look for what forms are there.

Similarly in #19 following, requiring coeff of x^4 in $(1 + 2x + 3x^2 + \cdots)^{-1/2}$, he follows a long, messy combinatoric soln with the remark:

In fact we require the coefficient of x^4 in the expansion of $((1-x)^{-2})^{-1/2}$, that is the expansion of $1 - x$; and thus the result should be zero.

The lesson being: Look first at the underlying form of number.

Inequalities

32. If a and x both lie between 0 and 1, then $(1 - a^x)/(1 - a) > x$.

Solution:

This is solved in the Algebra. We may also proceed thus:
We have to show that $1 + ax > x + a^x$
Put p/q for x, where p and q are integers
Thus we have to show that $(q - p + pa)/q > a^{p/q}$
Now this is obvious by Art. 681:
[arithmetical mean, any number of positive quantities > geometrical mean]:
for on the left-hand side we have the arithmetical mean of q quantities,
p of which are equal to a, and the rest equal to unity;
and on the right-hand side we have the geometrical mean
of the same q quantities.

* * *

We hardly consider, if we even learn, the forms of A.P and G.P and of the arithmetical mean and geometric mean. And yet these are forms of number. And when number in this form lies before us, our understanding of these forms and our ability to impose them on a problem consistent with them, simplifies our work.

Miscellaneous Examples

70. Solve (x - 2)(x - 3) = (155·78)/77²

Solution:

This equation can be solved in the ordinary way. Or we may write it thus:

$$(x - 2)(x - 3) = (2 + 1/77)(1 + 1/77)$$

and thus it is obvious that one solution is given by

$$x - 2 = 2 + 1/77 \text{ that is } x = 4^{1}/_{77}$$

And as the sum of the roots is 5, the other root is $^{76}/_{77}$.

* * *

This requires actually looking at the RHS of the equation and everything else follows from a simple knowledge of quadratics.

85. Solve $\dfrac{4x^3 + 4x^2 + 8x + 1}{2x^2 + 2x + 3} = \dfrac{2x^2 + 2x + 1}{x + 1}$

Solution:

Clear of fractions. Or we may proceed thus:

$$2x + \frac{2x + 1}{2x^2 + 2x + 3} = 2x + \frac{1}{x + 1}.$$

Therefore $\dfrac{2x + 1}{2x^2 + 2x + 3} = \dfrac{1}{x + 1}.$

Therefore $(x + 1)(2x + 1) = 2x^2 + 2x + 3$; &c.

* * *

Always the question is, "What forms are latent in the problem and what forms can be imposed upon it?"

181. Solve $9x^2 + 4x^3 = 1 + 12x^4$

Solution:

It is obvious that $x = 1$ is a root of this equation.
And we may write the equation thus: $9x^2 - 1 = 4x^3(3x - 1)$
that is, $(3x - 1)(3x + 1) = 4x^3(3x - 1)$
so that another root is given by $3x - 1 = 0$.
Hence, if we bring all the terms to one side we are sure that
$(x - 1)(3x - 1)$ will divide the terms.
On trial, we find that the equation may be written thus:
$(x - 1)(3x - 1)(2x + 1)^2$, so that the roots are 1, ⅓, -½.

* * *

First, is it obvious to you that $x = 1$ is a root? Do you look at an equation at this level first to see if there is an "obvious" root? And secondly, do you consider what can be done to change the problem to a simpler form? Here, the form is again $(a + b)(a - b) = a^2 - b^2$.

Fundamental Forms

Here are three examples of fundamental algebraic formulae from the textbooks of De Morgan, Todhunter, and Chrystal. These are also called **identities** as the LHS tautologically equals the RHS in each equation. These are, so to speak, the beginnings of the study of the form of number in its algebraic form.

De Morgan

This is the first list of identities in De Morgan's *The Study and Difficulties of Mathematics* (p. 87) which he wrote as an introduction to mathematics for younger students and is written at a lower level than his *Elements* series of pre-college texts.

$$(a + b) + (a - b) = 2a$$

$$(a + b) - (a - b) = 2b$$

$$a - (a - b) = b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(2ax + b)^2 = 4a^2x^2 + 4abx + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(x - a)(x - b) = x^2 - (a + b)x + ab$$

$$a/b = ma/mb$$

$$a + c/d = (ad + c)/d$$

$$a - c/d = (ad - c)/d$$

$$a/b + c/d = (ad + bc)/bd$$

$$a/b - c/d = (ad - bc)/bd$$

$$a/b \times c = ac/b = a/(b/c)$$

$$a/b \times c/d = ac/bd$$

$$(a/b)/c = a/bc = (a/c)/b$$

$$(a/b)/(c/d) = ad/bc = (a/c)/(b/d)$$

$$1/(a/b) = b/a$$

Todhunter

This is the first list of identities from Todhunter's *Algebra for the Use of Colleges and Schools* (p. 18ff).

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$

$(a + b)(a^2 - ab + b^2) = a^2 + b^2$

$(a - b)(a^2 - ab + b^2) = a^2 - b^2$

$(a + b)^3 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^3 + b^3$

$(a - b)^3 = (a - b)(a^2 - 2ab + b^2) = a^3 - 3a^2b + 3ab^3 - b^3$

$(b + c)(c + a)(a + b) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$

$(b - c)(c - a)(a - b) = a^2(b - c) + b^2(c - a) + c^2(a - b)$

$(a + b + c)(bc + ca + ab) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc$

$(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3abc$

$(b + c - a)(c + a - b)(a + b - c) =$

$a^2(b + c) + b^2(c + a) + c^2(a + b) - a^2 - b^2 - c^2 - 2abc$

$(a + b + c)^3 = a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3$

$= a^3 + 3a^2(b + c) + 3a(b^2 + 2bc + c^2) + b^3 + 3b^2c + 3bc^2 + c^3$

$= a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(c + a) + 3c^2(a + b) + 6abc$

Chrystal

And this is the table of identities from Chrystal's *Algebra - An Elementary Text Book* (p. 79ff). His table is much more thorough and complex.

I.
$(x + a)(x + b) = x^2 + (a + b)x + ab$
$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (bc + ca + ab)x + abc$
and generally
$(x + a_1)(x + a_2) \cdots (x + a_n) = x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_{n-1}x + P_n$
[where P_i = sum of "n choose i of the a's"]
$(x \pm y)^2 = x^2 \pm 2xy + y^2$
$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$ &c.;
the numerical coefficients being taken from the following table of binomial coefficients:

II.

Power	Coefficients												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

III.
$(x \pm y)^2 \mp 4xy = (x \mp y)^2$

IV.
$(x + y)(x - y) = x^2 - y^2$
$(x \pm y)(x^2 \mp xy + y^2) = x^3 \pm y^3$
and generally
$(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = x^n - y^n$
$(x + y)(x^{n-1} - x^{n-2}y + \cdots \mp xy^{n-2} \pm y^{n-1}) = x^n \pm y^n$
upper or lower sign according as n is odd or even.

V.

$$(x^2 + y^2)(x'^2 + y'^2) = (xx' \mp yy')^2 + (xy' \pm x'y)^2$$

$$(x^2 - y^2)(x'^2 - y'^2) = (xx' \pm yy')^2 - (xy' \pm x'y)^2$$

$$(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) = (xx' + yy' + zz')^2 + (yz' - yz')^2 + (zx' - z'x)^2 + (xy' - x'y)^2$$

$$(x^2 + y^2 + z^2 + u^2)(x'^2 + y'^2 + z'^2 + u'^2) = (xx' + yy' + zz' + uu')^2 + (xy' - yx' + zu' - uz')^2 + (xz' - yu' - zx' + uy')^2 + (xu' + yz' - zy' - ux')^2$$

VI.

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4$$

VII.

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

and generally

$(a_1 + a_2 + \cdots + a_n)^2$ = sum of the squares of $a_1, a_2, \ldots, a_n$ + twice sum of all partial products two and two.

VIII.

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 3a^2b + 3ab^2 + 6abc$$
$$= a^3 + b^3 + c^3 + 3bc(b + c) + 3ca(c + a) + 3ab(a + b) + 6abc$$

IX.

$$(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3abc$$

X.

$$(b - c)(c - a)(a - b) = -a^2(b - c) - b^2(c - a) - c^2(a - b)$$
$$= -bc(b - c) - ca(c - a) - ab(a - b)$$
$$= bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b$$

XI.

$$(b + c)(c + a)(a + b) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$$
$$= bc(b + c) + ca(c + a) = ab(a + b) + 2abc$$
$$= bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc$$

XII.

$$(a + b + c)(a^2 + b^2 + c^2) = bc(b + c) + ca(c + a) + ab(a + b) + a^3 + b^3 + c^3$$

XIII.

$$(a + b + c)(bc + ca + ab) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc$$

XIV.

$(b + c - a)(c + a - b)(a + b - c) =$

$$a^2(b + c) + b^2(c + a) + c^2(a + b) - a^2 - b^2 - c^2 - 2abc$$

XV.

$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 2b^2c^2 + 2a^2c^2 + 2a^2b^2 - a^4 - b^4 - c^4$

XVI.

$(b - c) + (c - a) + (a - b) = 0$

$a(b - c) + b(c - a) + c(a - b) = 0$

$(b + c)(b - c) + (c + a)(c - a) + (a + b)(a - b) = 0$

Afterword

I'm sorry that there were not more of Todhunter's *or thus* solutions. I have included all but one, which was a bit beyond the level of this text. Todhunter generally teaches the form of number directly, as he should. But it would be nice to have a textbook that concentrated on the multiplicity of the form of number in algebra. Perhaps some day I can write one.

While memorizing things is not the point of this work, there is a place for memory and its development in mathematics. I will close with a quotation by Jeudwine from the back of his *The Manufacture of Historical Materials*. Let us take his use of the word "man" as "mankind" to give it a broader sense.

The value of a trained memory, and the extraordinary powers of the memory when trained, are not sufficiently appreciated in our day. Our education, even in mathematics, is carried on so entirely by the use of books and paper that little encouragement is given to the balancing of facts or figures accumulated in the mind, leading to an indigestion of food which may be followed by terrible results.

The value of the training of the memory does not merely consist in the housing of the material; the process of digestion is of far greater importance, and one which has a far more enduring effect that the mere acceptance of facts by reading. It is this which gives the great value to the training of the memory; the man can turn over the facts which he has remembered; they are his own and not another's; he can consider for himself their explanation; he can see them in their full bearing, and not merely as the written, perhaps casual, opinion of one party, to whose work, if he has relied on the writing, he must refer before he can so consider them; and by the more frequent use of his memory he can strengthen the powers of observation, the accuracy of thought, the grasp in the mind of the bearing of the facts digested.