Everymind's
Second Circle of Algebra

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Published 13dec2019
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Dedication

This text is dedicated to

George Chrystal
Robert Murphy
Charles James Hargreave

for having respectively written

Algebra - An Elementary Text Book
A Treatise on the Theory of Algebraical Equations
The Resolution of Algebraic Equations

from which I have taken
the bulk of this text.

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Preface

With Euclid I-VI and De Morgan's Elements, we have the core of mathematics. This unit measure of knowledge begins our understanding of Geometry, Arithmetic, Algebra, Trigonometry, and Calculus. A harmonious beginning. An inner circle. In this text, we expand algebra into its second circle. This text assumes only the knowledge of this core, which is freely available in *Everymind's Euclid* (Euclid) and *Everymind's De Morgan's Elements* (DME).

Picking up where De Morgan left us, we investigate the expansion of his basic algebraic ideas to see how they develop and what realms of mathematics they begin to coalesce into. This is basic or fundamental algebra. In the 20th century, it collapsed into a brief study in high school and is even now hardly mentioned after that. The sources of this text are 19th century works. And in those days fundamental algebra filled large, even two-volume, texts. Bit and pieces of these large algebras turn up in the first chapters of modern university texts. But these fragmentary glimpses ignore the coherent study of these ideas as an organic whole.

Here I am trying to bring fundamental algebra and the development of its ideas back together so that continuities of thought across the various usages can again be seen. And this is an algebra that you can do and not simply prove. If *we learn mathematics by doing mathematics*, here are some mathematics to be done.

Introduction

Once again I am being selfish. I am expanding my understanding of fundamental algebra for my own benefit. And once again, I am sharing it with anyone who is interested. I've been over this algebraic ground three times already, filling notebooks with careful notes. This is my final pass over the fundaments of algebra and I'm using Chrystal's text mentioned in the dedication. Its twelve hundred pages should supply sufficient weight to fix an indelible impression upon my thick and math-resistant mind. The Murphy text comes recommended by Isaac Todhunter and is a fourth pass on what used to be called Theory of Equations. The Hargreave comes recommended by Todhunter and by George Salmon, who initially objected to Hargreave's work and was later won over to the extent of publishing it when Hargreave died.

Selfishly, I will only be expanding those ideas of algebra I actually care about. For example, there will be no Interest, Annuities, or Probability here. There will be Combinatorics, which is more or less Probability with word problems having nothing to do with chaos, because, as much as I dislike Combinatorics, many of my interests are built in part upon it. I will also forego Chrystal's development of complex numbers in trigonometric series as these will go into *Everymind's Second Circle of Trigonometry.* But most of fundamental algebra is here. If anything, I care about too much.

And selfishly, I will be really drilling down on my interests: Partial Fractions, Continued Fractions, Euclid's Algorithm, anything having to do with division. As Halsted said, *From fractions we get division*. You may have noticed in DME that Addition, Subtraction, and Multiplication took up only eleven pages between them while Division and Fractions took up twenty-five. That wasn't planned. Division is the decisive operator. Only in writing this paragraph did I realize my interests were universally divisive. That explains some other things about me, I'm sure.

You will find very little original in this text. I'm a harmonizer, not a creator. Any originality will be in the harmonization and organization of the content. As I do this, disparate things will come together. And these contrasts suggest other connections which you may not have made before either. And my continually going over the same ground finally exposes relations and techniques which were not obvious to me at first, even simple ones. Possibly, I am the last one to notice the obvious. But just in case I'm not the only slow&thick one present, I will share even these simple things with you.

I am finding that the best way to study mathematics is to write a book. Using only personal notebooks, I can tell myself that I understand many things I do not actually understand. But if, after making the notebooks, I pull it all together for general consumption, I am much more thorough in examining my understanding. So most of what is contained in this text will actually be correct. Well, everything should be correct. But if I mess up anywhere, I'm happy to take the blame. BTW, quotations from other works will be in italics. And I don't do footnotes.

1. Fundaments

The Form of Number

The first thing I want to do is to give you something of an understanding of my approach to mathematics. In Chrystal's preface, he writes:

Thus it becomes necessary, if Algebra is to be anything more than a mere bundle of unconnected rules, to lay down generally the three fundamental laws of the subject, and to proceed deductively -- in short, to introduce the idea of ***Algebraic Form****, which is the foundation of all the modern developments of Algebra and the secret of analytical geometry, the most beautiful of all its applications.*

De Morgan, in *his* Algebra, emphasizes algebra as a language and develops its meaningful expression. Todhunter, in *his* Algebra, emphasizes both the proof and the use of this form of number. Chrystal emphasizes form, the fundamental forms, of algebraic expression. And this is why I am working through his text and sharing it with you.

I am instinctively, intuitively, drawn to form. Fundamental forms. The multiplicity of a form. Expressions of the same form through different mathematics. It seems almost the most important facet of mathematics to grasp, this idea of form and the power it gives to our expression.

Formalism

Chrystal was writing in 1886 and in his writing you can see the approach of ideas, which by the 1920s would consume mathematics in its investigations of its own foundations. There would be three approaches -- Logicism, Formalism, and Intuitionism -- of which Formalism would come to dominate. Let me show you a glimpse of Formalism from Chrystal's early chapters. The three fundamental laws he mentions above are the Associative, Commutative, and Distributive Laws and he develops them in a kind of proto-formalism. As he does so, he develops the Laws of Signs. Here is a bit of that:

+(+a) = +a +(-a) = -a -(+a) = -a -(-a) = +a	×(×a) = (×a) ÷(×a) = ÷a ×(÷a) = ÷a ÷(÷a) = ×a
+a × +c = +ac -a × +c = -ac +a × -c = -ac -a × -c = + ac	+a ÷ +c = +(a÷c) -a ÷ +c = -(a÷c) +a ÷ -c = -(a÷c) -a ÷ -c = +(a÷c)

From these and similar, he shows that addition and subtraction are formally identical to multiplication and division. There is, he says, no distinction except the symbols, *a conclusion at first sight a little startling*, which does not extend to the Distributive Law. Nor, we might add, does it extend to the use of these two categories of operators, nor to all the forms they produce. They are at some, obviously abstract, level equivalent. But our practice, the mathematics we actually do, is untouched by this equivalency.

I don't really want to get bogged down discussing formalism or in what, in spite of its actual failure, it produced. We continue to live with what it produced -- and with what is still produces. But that's not where I live. Hermites, a profound algebraist, was only interested in general theorems when they could be used to solve particular problems. I think he sensed that an abstract mathematic had a tendency to go off on its own, saying true, yet often inapplicable or insignificant, things. Hermites's research was grounded on mathematics he could do. So he anchored his abstractions in the concrete. **That** is precisely my approach.

By "concrete" I absolutely do not mean "applied" as in "applied mathematics." There is no such thing as applied mathematics. There is only one mathematics. And it can be applied to the world of experience or to the world of mathematics or tautologically to itself. That last is what I believe Hermites wanted to avoid. I avoid that as well.

My problem with abstract mathematics is this: they give us almost no mathematics to do. I'm not criticizing abstract mathematics here. I'm just disappointed in them so far. I have worked through three books on Galois Theory and they basically amount to signage, posted on the border of fifth degree equations, saying, "Here there be no solutions by radicals." Most people seem to take them as saying, "I'd turn back if I were you," and they obediently go back to where they came from.

My response is more: "Okay, not by radicals. What else ya got?" And the books only shrug. Well, Hargreave did more than shrug. But I didn't know that as I was writing this. His work is the Easter egg in this book. At any rate, polynomials and their roots are expressions of laws governing the form of number. There is always a multiplicity of form. And the roads leading to the laws are infinite like the consciousness -- ours -- they spring from. When we arrive at our understanding, the polynomials must give up their roots.

So with this search for understanding, I am investigating the form of algebra, of number, in the world of mathematics.

Meaning

In both De Morgan and Chrystal, we find, explicit, *the assumption that there is always an answer to our question*. Isn't that extraordinary? If we have a mathematical question, we are told to assume that there is **always** an answer.

Now the answer may be nonsense. But in that case, we assume the nonsense to be our own fault. We either set up the problem incorrectly or we are prematurely asking the question. And if we discover that the nonsense is not our fault, then mathematics is always gradually expanded as the nonsense is given meaning. So the answer actually was there all along. There is no nonsense in mathematics.

As for the fundamental laws, we've seen them proved for addition with methods like:

$$\begin{aligned} &3 + 4 \\ &= (1 + 1 + 1) + (1 + 1 + 1 + 1) \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ &= (1 + 1 + 1 + 1) + (1 + 1 + 1) \\ &= 4 + 3 \end{aligned}$$

And for multiplication, we've seen it done with rectangles of dots. Reading Chrystal, it occurred to me that by using Euclid 2.1 and its "corollaries" of immediately succeeding propositions, you can prove the three laws in multiplication with rectangles. The Commutative Law is "a rectangle is indifferent to its orientation." The Associative Law becomes "a rectangle is indifferent to the method of its rectangular division." And Euclid 2.1 does it for the Distributive Law.

As Chrystal works through the basic three laws, we see that negative numbers still retain for him a cachet and are not yet formalized into blandness. Here, "**-a**" is explained as ***a*** *to be subtracted*, in other words, algebra, as general theory, always implies practical use. He points out that "1/2 × 2/3 = 1/3" is an operation while "1/3 is 1/2 of 2/3" is an interpretation.

Chrystal defines "equal to" as "transformable by the fundamental laws of algebra into." Note that nothing is equal to itself; everything is identical to itself, which is not the same thing.

He defines zero as *the limit of the difference of two quantities that have been made to differ as little as we please*. Or:

$$(a + x) - a = a - a + x = x$$

which has the limit 0 as $(a + x) \to a$. I find it charming that, even after Weierstrass has fixed the definition of the limit of increasing and decreasing quantities, Chrystal continues to take his pleasure with the large and small. He defines unity as a ratio of similarly diminishing difference, where

$$(a + x) \div a = a \div a + x \div a = 1 + x \div a$$

has the limit of unity as $x \to 0$.

There is for Chrystal a symmetry between one and zero:

$+ a - a = - a + a = 0$	$\times a \div a = \div a \times a = 1$
$a + 0 = a - 0$ $\therefore + 0 = - 0$	$a \times 1 = a \div 1$ $\therefore 1 \times 1 = 1 \div 1$

And indeed, part of recognizing the form of number is seeing precisely what is under your nose, even in the cases of ones and zeroes, before you start taking it apart with your tools of operation. If you have

$$9x^3 + 4x = 3x^4 + 10x^2$$

you should first see that 0 and 1 are roots. Then you can go looking for the other two. I had not seen this expression before either:

$$b \times (a \times \text{bth root of unity}) = a$$

Everyone else has simply said: $b \times (a \times 1/b) = a$, which is far less interesting. Chrystal also points out, in this first chapter of his, the **Principle of Substitution** which is painfully obvious to read about. I'll use a painfully obvious example.

If you have a true statement in algebra

$$a(bc) = (ab)c$$

then you can choose any expression for a, b, and c

$$a = x/y \quad b = x^2 \quad c = y^2x$$

and the truth remains

$$x/y(x^2 \cdot y^2x) = x^4y = (x/y \cdot x^2)y^2x$$

And, yes, we knew all this. But we do not all take all the of advantage of it. Later, here and there, I will point out where you may not be taking advantage of the Principle of Substitution. And if you are slow&thick like I am, you will wonder, as I did, how you overlooked it.

Euclid's Algorithm

In DME I called Euclid's Algorithm the determining of the Greatest Common Factor or GCF. I did that out of politeness towards other widely-used text books. But I don't like calling it that. I like tradition and traditionally Euclid's Algorithm determines the Greatest Common Measure or GCM, as Euclid did not have factors (or divisors as in

GCD). So we're going with GCM, denoted gcm(a,b) = c, a,b,c∈**N**. There, I feel better.

In mathematics, everything interesting begins with division. When you determine the GCM, you can take the multiple of the divisor either above or below the dividend and the remainder with the smallest absolute value will shorten the process.

1595)4323(2	1595)4323(3
1133)1595(1	462)1595(3
462)1133(2	209)462(2
209)462(2	44)209(5
44)209(4	11)44(4
33)44(1	0
11)33(3	
0	

RHS shorter by two steps. The standpoint you need to take here is not "I am learning algebra" but "I am becoming an algebraist." You are developing your own mind. And so with each new idea, you must ask yourself, "How can I bring this idea to bear on my practice of algebra?" And my approach, as I work through Chrystal, is to grab every idea which seems to relate to my practice and interests. And this serves as a filter to bring more useful ideas into my understanding.

Recall our notation for "a and b are prime to each other": p(a,b) and our use of "a divby b" for "a is divisible by b without remainder." Back to the GCM. All that follows will eventually apply to integral functions.

Thm. 1.1. p(a,b), ∀h∈**N** ⇒ any common factor of a·h and b divides h exactly.
Cor 1. p(a,b), a·h divby b ⇒ h divby b
Cor 2. p(a, [b, c, d, ...]) ⇒ p(a, b·c·d···)
Cor 3. A = {a_1, a_2, ...} B = {b_1, b_2, ...} if p(A,B) ⇒ p($\prod a_i$,$\prod b_i$)
Cor 4. p(a,b), ∀m,n∈**N** ⇒ p(a^m,b^n)

Also recall that I only do proofs when they are the best explanation. But as you actually have the core of mathematics through De Morgan and Euclid, you have done far too many proofs to need me to do any easy ones for you. If you have any doubts as to my assertions or feel the need, as you sometimes should, for a proof I haven't supplied, then go do one and, whether you succeed or not, go find one that satisfies you in order to check your work. And it is important that you do find a proof actually satisfying. Look for ones that do this. Cherish your own mind.

Thm. 1.2. Represent $a \in \mathbf{N}$ as $a = qb + r$: $r<b \Rightarrow$ as r is pos/neg $\exists!$ representation.
[Represent natural number a as $q \cdot b + r$ such that r is less than b then as r is positive or negative there exists a unique q,b, and r.]
Proof
1) r > 0
q max of $qb < a \Rightarrow a - qb = r$: $r < b \therefore a = qb + r$ unique
Else $xb + p = a$ where $x \neq q$
$\therefore qb + r = xb + p \therefore (r - p) = (q - x)b \therefore (r - p)$ divby b ↴ (r,p < b by hyp.)
2) r < 0 Sym. proof. ■
Cor. 1. a divby/!divby b as least remainder r does/doesn't vanish.

Above, the r's are minimum (min) pos, neg remainders. Min positive is usually assumed. If the restriction of r < b is removed then r is called the **residue**.

Thm. 1.3. remainder r for a,a' equal with respect to (wrt) b $\Leftrightarrow$ (a - a') divby b
Proof
$a = rb + t$ $a' = sb + t \therefore (a - a') = (r - s)b + (t - t)$ ■
Example
$a = 29$ $a' = 13$ $b = 4$
$a = 7 \cdot 4 + 1$ $b = 3 \cdot 4 + 1$ $(a - a') = 4 \cdot 4 = 16$ divby b

We saw in DME that polynomials with integer coefficients (coeffs) are subject to Euclid's Algorithm. Therefore they are "as integers" and what applies to integers in arithmetic and number theory must apply to these integral functions (ifn). The following theorems can all be proven from the form of the GCM.

Thm. 1.4. In gcm(a,b), ∀remainder has form: $\pm(Aa - Bb)$ positive (pos) on odd steps, negative (neg) on even, where $A,B \in \mathbf{N}$ take on successive values.
Cor. 1. $\text{gcm}(a,b) = g \Rightarrow \pm g = (Aa - Bb)$
Cor. 2. $\text{gcm}(a,b) = 1 \Rightarrow \pm 1 = (Aa - Bb)$
Note: In each case if $p(A,B) \Rightarrow a/g, b/g \in \mathbf{N} = l,m \therefore 1 = \pm(Aa - Bb)$
Else if !p(A,B) then common factor C $\therefore$ 1 divby C↴
All of this pertains to continued fractions and Cor. 2 will be seen again in partial fractions of fractions of ifns.
Cor. 3. ∀common factor of a,b is a factor of gcm(a,b).
Cor. 4. To find gcm(a,b,c,…): gcm(a,b) = a', gcm(a',c) = b', …

To see that A,B take on successive values, let's see where they come from.
Here, in the GCM algorithm, c,d,e are successive remainders:
If $a = pb + c$ then $c = +(a - pb)$
Next step, $d = b + qc = b - q(a - pb) = -(qa - (1 + pq)b)$
$e = c - rd = (a - pb) + r(qa - (1 + pq)b) = +((1 + qr)a - (p + r + pqr)b)$

Examples

1) Express gcm(565,60) as A·565 - B·60

565 = 9·60 + 25

60 = 2·25 + 10

25 = 2·10 + 5

10 = 2·5

∴ 25 = 565 - 9·60

10 = 60 - 2(2·565 - 9·60) = -(2·565 - 19·60)

5 = 25 - 2·10 = 565 - 9·60 + 2(2·565 - 19·60) = 5·565 - 47·60

2) Solve 5A - 7B = 1

7 = 1·5 + 2

5 = 2·2 + 1

∴ 2 = 7 - 5   1 = 5 - 2(7 - 5) = 3·5 - 2·7

∴ A = 3, B = 2

Thm. 1.5. Given fraction p/ab: p < ab and p(a,b) ⇒ ∃! representation a'/a + b'/b - k: a',b' > 0, a',b' < a,b, k = 0∨1 (∨ ≡"or") as k is/isn't integral part of a'/a + b'/b.

Example

6/35: 35 = 5·7

3·5 - 2·7 = 1   (by above example #2)

∴ 6/35 = 6/35·(3·5 - 2·7) = 18/7 - 12/5

   = (2·7 + 4)/7 - (3·5 - 3)/5 = $2^4/_7 - 3^3/_5$ = 3/5 + 4/7 - 1

Proof

Aa - Bb = ±1

±p/b·A ∓ p/a·B = p/ab   (× ±p/ab)   [1]

upper sign:

pA = lb + b'   pB = ma - a' where 0 < a',b' < a,b

∴ p/ab = l - m + a'/a + b'/b   (by [1])

p/ab proper fraction ∴ integral part RHS = 0

integral part of a'/a + b'/b ≤ 1

∴ l - m = 0 ∨ -1

lower sign:

Sym. with pA = lb - b'   pB = ma + a'

I'm not sure why Chrystal includes this next theorem where he does. Maybe he wants to reassure us that we won't run out. I include it for a study of simplicity in proofs.

Thm. 1.6. The number of prime integers is infinite.

Proof

Else last prime = p

Let P = ∏(primes ≤ p) ∴ P divby ∀primes ≤ p

∴ P+1 !divby ∀prime ≤ p ↴

∴ primes infinite

Form of Division

If something is true of $n \in \mathbf{N}$, then it is true of $\mathbf{Z}$ and with due care it is true of integral functions (ifns). Let's divide 719 by 2,3,4… until we run out of dividend.

```
2) 719 (
3) 359 ( r 1
4) 119 ( r 2
5)  29 ( r 3
     5   r 4
```

$\therefore 719 = 1 + 2 \cdot 2 + 3 \cdot 2 \cdot 3 + 4 \cdot 2 \cdot 3 \cdot 4 + 5 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

Take a minute to figure out what we are doing here. It's a bit weird. Now divide 719 by 2,4,6,…

```
2) 719 (
4) 359 ( r 1
6)  89 ( r 3
8)  14 ( r 5
     1  r 6
```

$\therefore 719 = 1 + 3 \cdot 2 + 5 \cdot 4 \cdot 2 + 6 \cdot 6 \cdot 4 \cdot 2 + 1 \cdot 2 \cdot 4 \cdot 6 \cdot 8$

If we call 2,3,4,… or 2,4,6,… r_i [1-n] and call 1,2,3,4,5 or 1,3,5,6,1 p_i [0-n] and call 719 N, then in general:

Thm. 1.7. Let r_i denote any series of $n \in \mathbf{N}$ in no way restricted, then any $N \in \mathbf{N}$ has form:

$$N = p_0 + p_1 r_1 + p_2 r_1 r_2 + \cdots + p_n r_1 r_2 \cdots r_n \qquad [1]$$

where $p_i < r_{i+1}$ and given the r_i, the result is unique.

Proof

If we designate the successive dividends N_i [1-n]

$$N = p_0 + N_1 r_1 \qquad p_0 < r_1$$
$$N_1 = p_1 + N_2 r_2 \qquad p_1 < r_2$$
$$\cdots$$
$$N_{n-1} = p_{n-1} + N_n r_n$$

And by subbing

$$N = p_0 + r_1(p_1 + N_2 r_2)$$
$$= p_0 + p_1 r_1 + r_1 r_2 N_2$$

and so on until we have N in form [1] and form unique.
Else $\exists p'_i \neq p_i$ then set our variant results equal and divide by p_1
$p_0/r_1 + (p_1 + p_2 r_2 + \cdots) = p'_0 + (p'_1 + p'_2 r_2 + \cdots)$
Terms in parens are in $\mathbf{N}$ and fractions are proper.
$\therefore p_0/r_1 = p'_0/r1 \therefore p_0 = p'_0$ and sums in parens are equal.
Sym. true of all divisions by r_i ■

Let's look at fractions this way, beginning with a general theorem.

Thm. 1.8. Any proper fraction A/B has form

$$A/B = p_1/r_1 + p_2/r_1r_2 + p_3/r_1r_2r_3 + \cdots + p_n/r_1r_2\cdots r_n + F$$

where $p_i < r_i$ and F is either zero or has a limit of zero as $n\to\infty$. Given r_i, form is unique.

Proof

You do it. $A/B = Ar_1/Br_1 = (Ar_1/B)/r_1 = (p_1 + q_1/B)/r_1 = p_1/r_1 + 1/r_1 \cdot q_1/B$ etc.

If A/B in lowest terms, F = 0 when $\prod r_i = mB$ for some $m\in\mathbf{N}$

Examples

1) A/B = 444/576

$576 = 2^6\cdot 3^2$ $\quad r_i = 2,4,6,\ldots \Rightarrow 444/576 = 37/48 \therefore \prod r_i = 48$

$\therefore 444/576 = 1/2 + 2/2\cdot 4 + 1/2\cdot 4\cdot 6$

2) A/B = 11/13 $\quad !\exists \prod r_i = 13 \quad r_i = 2,3,4,\ldots$

$11/13 = 1/2 + 2/2\cdot 3 + 0/2\cdot 3\cdot 4 + 1/2\cdot 3\cdot 4\cdot 5 + 3/2\cdot 3\cdot 4\cdot 5\cdot 6 + 3/2\cdots 6\cdot 13$

Here series cannot terminate. So we terminate F with the factor 13 in the denominator (denom).

Here is an algorithm for expanding a proper fraction with a series of proper fractions with numerators of unity. Again we take 11/13 and each multiplier is minimal.

```
      11
       2 ×
13 ) 22 ( 1
     13
      9
      2×
13 ) 18 ( 1
     13
      5 ×3
13 ) 15 ( 1
     13
      2 ×7
13 ) 14 ( 1
     13
      1 ×13
13 ) 13 ( 1
      0
```

$\therefore 11/13 = 1/2 + 1/2\cdot 2 + 1/3\cdot 2\cdot 2 + 1/7\cdot 3\cdot 2\cdot 2 + 1/13\cdot 7\cdot 3\cdot 2\cdot 2$

By using a minimal multiple, remainders go to zero while the multipliers increase.

```
2/3  =   2 ×2  = 1/2 + 1/3
     3 ) 4 ( 1
         3
         1 ×3
     3 ) 3 ( 1
         0
```

But use multipliers 2,4,4,… then

$2/3 = 1/2 + 1/2\cdot 4 + 1/2\cdot 4^2 + \cdots + 1/2\cdot 4^n + 1/2\cdot 4^n\cdot 3$

where you have to terminate F.

Recall from DME that by using the same divisor N there are N-1 different remainders and all results that hold for repeating decimals holds here.

Thm. 1.9. $\forall N \in \mathbf{N}$ has unique form

$$N = p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0$$

where $\forall p_i < r$.

Proof follows from Thm. 1.8. And this is nothing but positional notation. If we divide N by r the remainder is p_0, divide N_1 by r the remainder is p_1, and so on. Process terminates with $0 \le p_n < r$. And we have seen this next bit before:

Base 10	**Base 6**
10) 719 (	6) 719 (
10) 71 (r 9	6) 119 (r 5
7 (r 1	6) 19 (r 5
$719 = 7\cdot10^2 + 1\cdot10 + 9$	3 (r 1
	$719_6 = 3\cdot6^3 + 1\cdot6^2 + 5\cdot6 + 5$

$$\therefore 719_{10} = 3155_6$$

Thm. 1.10 Any proper fraction A/B has the form

$$A/B = p_1/r + p_2/r^2 + p_3/r^3 + \cdots + p_n/r^n + F$$

where $\forall p_i < r$ and either $F = 0$ or $F \to 0$ and $n \to \infty$. Proof follows from Thm. 1.9. And this is only decimal fractional notation. If r is the base and F = 0 then

$$A/B = 0.p_1p_2p_3 \ldots p_n$$

From DME we know that $F = 0 \Leftrightarrow$ denom in form $2^m \cdot 5^n$ for base 10. So if you wanted 2/3 to 5 decimal places:

$$2/3 = (2\cdot10^5)/(3\cdot10^5) = (66666+2/3)/10^5 = 0.66666 + (2/3)/10^5 \cong 0.66666$$

And 5/64 must terminate as $64 = 2^6$:

$$(5\cdot10^6)/(64\cdot10^6) = 78125/10^6 = 0.078125$$
$$\text{or } (5\cdot5^6)/(64\cdot5^6) = 78125/10^6$$

Now this next bit is not exactly our division algorithm but it's close. And it shows that what we did with bases can be done with positional fractions.

Base 10	**Base 7**
0.168	0.168
10	7
1.) 68	1.) 176
10	7
6.) 80	1.) 232
10	7
8.) 00	1.) 624
$\therefore = 1/10 + 6/10^2 + 8/10^3$	7
	4.) 368
	7
	2.) 576

$\therefore 0.168_{10} \cong 0.11142_7$

We are looking for the form of number through the form of division in such a way that the base is arbitrary. All of this is true of all bases and all positional notation. Two more theorems. Consider that $\forall n \in \mathbf{N}$ has a different form in each base.

Thm. 1.11. $\forall$N in base r, divide N and $\sum$(digits of N) by r-1 $\Rightarrow$ same remainder in each case.

Proof

$N = p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0$

$N - (p_0 + p_1 + \cdots + p_n) = p_1(r - 1) + p_2(r^2 - 1) + \cdots + p_n(r^n - 1)$

$m \in \mathbf{N}$, $r^m - 1$ divby $r - 1$

$\therefore N - (p_0 + p_1 + \cdots + p_n) = \mu(r - 1)$ for some $\mu \in \mathbf{N}$ [1]

Let remainder of $N \div (r - 1)$ be s: $N = \lambda(r - 1) + s$

$\therefore$ by [1] $(p_0 + p_1 + \cdots + p_n) = (\lambda - \mu)(r - 1) + s$ ■

Cor. 1. Base 10, N÷9 (or 3) has same remainder as $\sum$(digits of N)÷9 (or 3)

Cor. 2. $n \in \mathbf{N}$, m = 3n or 9n, $\sum$(digits of m) = 3 or 9.

Thm. 1.12. Lambert's Theorem

Let q_i, r_i [1 - n] be quotients and remainders when B divided by A, r_1, r_2, ... then

$$A/B = 1/q_1 - 1/q_1 q_2 + 1/q_1 q_2 q_3 - \cdots + (-1)^{n-1}/\prod q_i + F$$

where $F = ((-1)^n r_n)/(\prod q_i \cdot B)$ $\therefore$ $F < 1/\prod q_i$

Proof

You do it. $B = Aq_1 + r_1$ $\therefore A/B = 1/q_1 - r_1/q_1 B$

$B = r_1 q_2 + r_2$ $\therefore A/B = 1/q_2 - r_2/q_2 B$

…

Then combine and unpack A/B in the form of the theorem.

Example

$113/244 = 1/2 - 1/2\cdot13 + 1/2\cdot13\cdot24 - 1/2\cdot13\cdot24\cdot61$

where $r_4 = 4$ $\therefore$ $4/244 = 1/61$

$\therefore$ sum of 1st three terms within 3d term (1/624) of 113/244

These are *almost* continued fractions, aren't they?

Ratio and Proportion

A few simple things as corollaries to what we already know. Let a > b, x > 0.

$$\frac{a+x}{b+x} - \frac{a}{b} = \frac{b(a+x) - a(b+x)}{b(b+x)} = \frac{x}{b} \cdot \frac{b-a}{b+x}$$

b - a < 0 ∴ RHT < 0 ∴ (a+x)/(b+x) < a/b (RHT is Right Hand Term)
Sym. a > b, a > 0

$$\frac{a-x}{b-x} - \frac{a}{b} = \frac{b(a-x) - a(b-x)}{b(b-x)} = \frac{x}{b} \cdot \frac{a-b}{b-x}$$

∴ (a - x)/(b - x) > a/b

If a:b::b:c then $a:c::a^2:b^2::b^2:c^2$ and $b = \sqrt{ac}$.
If a:b::b:c::c:d then $a:d::a^3:b^3::b^3:c^3::c^3:d^3$ and $b = \sqrt[3]{a^2d}$ $c = \sqrt[3]{ad^2}$.
These mean that inserting n mean proportions between any two values requires taking nth roots: 1:x::x:2 ∴ $x^2 = 2$ ∴ $x = +\sqrt{2}$.

We can interestingly extend Euclid 5.2 as follows:

Thm. 1.13. If $a_1:b_1::a_2:b_2:: \cdots ::a_n:b_n$ then ∀l this extends to
$:: \sqrt[r]{(l_1a_1^r + l_2a_2^r + \cdots + l_na_n^r)} : \sqrt[r]{(l_1b_1^r + l_2b_2^r + \cdots + l_nb_n^r)}$

Proof

a_i/b_i = some δ ∴ $a_i^r = (\delta b_i)^r = \delta^r b_i^r$ for ∀i
∴ $\sqrt[r]{(\sum l_ia_i^r)} = \sqrt[r]{(\delta^r \sum l_ib_i^r)} = \delta^r\sqrt{\sum l_ib_i^r}$
∴ $\sqrt[r]{(\sum l_ia_i^r)}/\sqrt[r]{(\sum l_ib_i^r)} = \delta = a_1/b_1 = a_2/b_2 = \cdots$ ■

More simply, if a:b::c:d then ∀l,m,p,q,r:

$$la+mb:pa+qb::lc+md:pc+qd \text{ and } la^r+mb^r:pa^r+qb^r::lc^r+md^r:pc^r+qd^r$$

which I leave for you to work out. The key ideas from Euclid 5.11 is that if a:b::c:d then a/b = c/d and this must equal something OR a/b = c/d = λ which can remain abstract if it wants to. And if a/b = c/d then we have a/b + 1 = c/d + 1 and one is λ/λ where λ is any algebraic term. It is easier to work things out with the form a/b than a:b in most cases.

We can extend Euclid 5.12 even further:

Thm. 1.14. ∀ homogeneous (homog) ifn $\varphi(x_1,x_2,\ldots,x_n)$ of r°
and $a_1:b_1::a_2:b_2::\cdots::a_n:b_n$ and $a_i/b_i = p$
⇒ $\varphi(a_1,a_1,\ldots,a_n):\varphi(b_1,b_2,\ldots,b_n) = p$

Examples

1) a:b::c:d, A:B::C:D ⇒ $a\sqrt{A} - b\sqrt{B} : c\sqrt{C} - d\sqrt{D} :: a\sqrt{A} + b\sqrt{B} : c\sqrt{C} + d\sqrt{D}$

Let a/b = c/d = λ A/B = C/D = μ ∴ a = λb c = λd A = μB C = μD

$$\therefore \frac{a\sqrt{A} - b\sqrt{B}}{c\sqrt{C} - d\sqrt{D}} = \frac{\lambda b\sqrt{\mu B} - b\sqrt{B}}{\lambda d\sqrt{\mu D} - d\sqrt{D}} = \frac{(\lambda\sqrt{\mu} - 1)b\sqrt{B}}{(\lambda\sqrt{\mu} - 1)d\sqrt{D}} = \frac{b\sqrt{B}}{d\sqrt{D}}$$

Sym. $(a\sqrt{A} + b\sqrt{B})/(c\sqrt{C} + d\sqrt{D}) = b\sqrt{B}/d\sqrt{D}$

2) x/(b+c-a) = y/(c+a-b) = z/(a+b-c) ⇒ (b-c)x + (c-a)y + (a-b)z = 0

Let [1] = p then x = (b+c-a)p   y = (c+a-b)p   z = (a+b-c)p

∴ (b-c)x + (c-a)y + (a-b)z = (b-c)(b+c-a)p + (c-a)(c+a-b)p + (a-b)(a+b-c)p

and algebrate.

3) Note that I use ·|·(x,y) to indicate "between x and y".

If b is mean proportion ·|·(a,c) then $(a+b+c)(a-b+c) = a^2 + b^2 + c^2$ [1]

a:b::b:c ∴ $b^2 = ac$ (hyp.)

$(a+c)^2 - b^2 = a^2 + b^2 + c^2$ ([1])

∴ $(a+c)^2 - b^2 = (a+c)^2 - ac = a^2 + c^2 + ac = a^2 + b^2 + c^2$

By setting a/b = b/c = p you can show $(a+b+c)^2 + a^2 + b^2 + c^2 = 2(a+b+c)(a+c)$

We should note that by choosing the unit, any two rational quantities in a ratio can be expressed as integers: 3 1/4 : 4 3/8 Let the unit be 1/8. Ratio becomes 26:35. In his demonstration that proportions hold for all numbers as well as magnitudes, Chrystal writes:

Any theory may be expressed in algebraical symbols, provided the fundamental principles of its logic are in agreement with the fundamental laws of algebraical operation.

That Euclid V applies to number is long established. If you are interested in that establishment, see De Morgan's *The Connexion of Number and Magnitude* for the prettiest development of these ideas.

Variance

We can relate proportions to functions as follows.

Let y = f(x) then x→y, x'→y' ∴ y : y' :: x : x'

If we take actual values for $x' = x_0$, $y' = y_0$ then $y : y_0 :: x : x_0$ or $y/y_0 = x/x_0$ and we arrive at $y = y_0/x_0 \cdot x$ or y = ax where $a = y_0/x_0$.

∴ y' = ax' ∴ y/y' = ax/ax' = x/x' ∴ y : y' :: x : x'

We say here that "y varies directly as x" or "y is proportional to x." For notation, we will use yRx. This can be expanded by replacing x as follows:

$y = ax^2$	$\Rightarrow$	$y : y' :: x^2 : x'^2$	
$y = a/x$	$\Rightarrow$	$y : y' :: 1/x : 1/x'$	[1]
$y = a/x^2$	$\Rightarrow$	$y : y' :: 1/x^2 : 1/x'^2$	[2]
$y = a(x+b)$	$\Rightarrow$	$y : y' :: x + b : x' + b$	

In [1], y varies "inversely as x" and in [2], "inversely as the square of x." Further:

$u = axy$	$\Rightarrow$	$u : u' :: xy : x'y'$
$u = a(x + y)$	$\Rightarrow$	$u : u' :: x+y : x'+y'$
$u = a \cdot x/y$	$\Rightarrow$	$u : u' :: x/y : x'/y'$

You can prove for yourself the following:

1. if zRy and yRx then zRx
2. if y_1Rx_1 and y_2Rx_2 then $y_1y_2Rx_1x_2$
3. if yRx then zyRzx whether z is variable or constant
4. if zRxy then xRz/y and yRz/x
5. if z depends only on x and y and zRx when y constant and zRy when x constant then zRxy when x,y both vary
6. if z depends only on $x_1, x_2, \ldots x_n$ and varies as any one of them when the rest are constant, then when all vary $zR\prod x_i$
7. if zRx (y constant) and zR1/y (x constant) then zRx/y (x,y vary)

Examples

1) ∀ΔABC, let A = area, b = base BC, t = altitude ⇒ ARb (t constant) and ARt (b constant) ∴ ARtba when t,b vary. And constant a = ½. But we knew that.

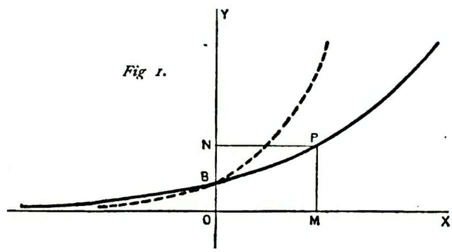
2) sRt^2 (f constant) sRf (t constant) 2s = f when t = 1 Required relation of s,f,t
By #5 above, $sRft^2$ ∴ $s = aft^2$ where a is some constant. We need a.
t = 1 s = f/2 ∴ $\frac{1}{2}f = af1^2$ ∴ ½f = af ∴ a = ½ ∴ $s = \frac{1}{2}ft^2$

Logarithms

Consider the exponential fn a^x where a∈**R** and a > 1. Let x∈**R**. If x ∈ **R-Q**, we can approximate a^x to any degree of accuracy with some m/n ∈ **Q**. So we can consider only such $a^{m/n}$. And if x<0, we define a^x as $1/a^{-x}$. Defined in this way, a^x has the range [0,∞). If y∈**R**, y > 1, n∈**N**, then as a > 1, $a^{1/n} > 1$ and as n→∞, $a^{1/n} \to 1$. Given n, we can choose m∈**N**: $a^{m/n}$ m→∞, $a^{m/n} \to \infty$. Then ∀y ∃n: $a^{1/n} < y$ ∴ $a^{m/n} < y < a^{(m+1)/n}$. The difference of these outside terms is $a^{m/n}(a^{1/n} - 1)$ ∴ n→∞ this difference→0. So no matter what y is, we can find an x: $a^x = y$. Now let y ∈ (0,1) then 1/y > 1 ∴ ∃x: $a^x = 1/y$ ∴ $a^{-x} = y$. So for any y ∈ (0,1) there is always an x: $a^x = y$ and, as no y is excluded, x is continuous. A quick table:

x	-∞	neg x	-1	0	1	pos x	+∞
$\mathbf{a^x}$	0	< 1	1/a	1	a	>1	+∞

In this figure, the solid curve is 10^x and the dotted one is 100^x. Note that both intersect the Y-axis at (0,1) and that for any x, the value of $100^x = 2 \cdot 10^x$.



In $y = a^x$, y is a continuous fn of x. But we can use the same fn to determine a continuous fn x of y. In a^x, a is the **base**. When y is a fn of x to base a then $y = a^x$. The inverse fn x(y) is the logarithm of y to base a: $x = \log_a y$. All properties of the log fn must be derivable from its inverse exponential fn, which is to say **from the laws of indices**. If we take "log" to mean "$\log_a$", it follows from the above that $y = a^{\log y}$. In all that follows, we will take "log" without a base to mean "$\log_{10}$" and we will use "ln" to mean "$\log_\varepsilon$".

Thm. 1.15. If 10 is the logarithm base, then the characteristic of the log depends on the position of the decimal point and the mantissa is independent of the decimal point, depending only on the succession of digits.
Proof
N is any number formed by a row of digits, c is N's characteristic, m N's mantissa. Then any number with the digits of N but with a different decimal point placement has the form $10^i \cdot N$ $i \in \mathbf{Z}$. Then $\log 10^i \cdot N = \log 10^i + \log N = i + \log N = (i+c).m$. But by hyp, $m \in \mathbf{Q}$, $c,i \in \mathbf{N}$ ∴ mantissa of $\log 10^i \cdot N = m$ and its characteristic is i+c. ∴ characteristic alone is altered by decimal point. ■

Thm. 1.16. If a series of numbers are in G.P., their logs are in A.P.
Proof
Take $y_1, y_2, \ldots, y_n$ in G.P. These equal $a^\alpha, a^{\alpha+\beta}, a^{\alpha+2\beta}, \ldots, a^{\alpha+(n-1)\beta}$.
∴ logs are $\alpha, \alpha+\beta, \alpha+2\beta, \ldots, \alpha+(n-1)\beta$. ■

Historically speaking, it was this relation that led to logarithms.

Thm. 1.17. Given a system of logs to base a, we can convert them to another base b by multiplying them by $1/\log_a b$.
Proof
$x = \log_b y \Rightarrow y = b^x$
∴ $\log_a y = \log_a b^x = x \log_a b$
∴ $\log_b y = x = \log_a y / \log_a b$ ■ [1]

Def. $\mu = 1/\log_a b$ is the **modulus** of system base b to system base a.

Cor. 1. If in [1], y = b then $\log_b a = 1/\log_a b$ ∴ $\log_a b \cdot \log_b a = 1$
Cor. 2. Let "log" be "$\log_a$" then $y = b^x$ can be written $y = a^{x \cdot \log b}$ or if "log" is "$\log_b$" then $y = a^{x/\log a}$ ∴ the graph of b^x is deducible from the graph of a^x by:

$$\text{abscissa } b^x : \text{abscissa } a^x :: 1 : \log_a b$$

In other words, given any two exponential graphs A,B, either A is the orthogonal projection of B or B of A onto a plane passing through the Y-axis.

Historically speaking, logarithms in tables were used for numerical calculation. So algebraic and other texts devoted considerable space to the optimized methods of calculation by logs and of log table usage. Now, the logs are built into the calculating device and we never see them. But here is an interesting idea. Recall from DME that if x increases incrementally {x, x+h, x+2h, ...} then, to a limited extent, f(x) will increase incrementally.

Let h be the difference between values in a log table.
Let D = f(a+h) - f(a) ≡ tabular distance of the logs.
Let a+h' be a value not in the table between a, a+h. such that h'<h.

$$\therefore \quad \frac{f(a+h') - f(a)}{D} = \frac{(a+h') - a}{(a+h) - a} = \frac{h'}{h}$$

In usage, f(a), D, and h are known. We now have a relation between h' and f(a+h'). So if we know h' or f(a+h') we have

$$f(a+h') = f(a) + h'/h \cdot D$$
$$a + h' = a + ((f(a+h') - f(a))/D) \cdot h$$

From the first, we find a value of f given an increment of a. From the second, we get an intermediate value of a from an increment of f. This was called the **Rule of Proportional Parts**.

Combinatorics

Let us deepen our language of permutations and combinations. Recall from DME that is we have m things and choose n, notation is $C_{m|n}$ for combinations (combs) and $P_{m|n}$ for permutations (perms). We will use Chrystal's language in what follows instead of De Morgan's but keep our notation.

Permutations

Thm. 1.18. The number of r-perms of n things ($P_{n|r}$) is

$$n(n-1)(n-2)\cdots(n-(r+1))$$

[proof follows]

Proof

Enumerate the r-perms of a_i [1-n] into classes as follows:

1st. All those where a_1 is first.

2d. All those where a_2 is first. And so on.

There are as many perms where a_1 is first as there are (r-1)perms of the remaining n-1 letters OR there are $P_{n-1|r-1}$ perms of the first class above.

This is true of all n classes.

∴ $P_{n|r} = nP_{n-1|r-1}$

If n≥r, this is true of ∀n,r ∈ **N**

∴ $P_{n|r} = nP_{n-1|r-1}$

$P_{n-1|r-1} = (n-1)P_{n-2|r-2}$

…

$P_{n-r+2|2} = (n-r+2)P_{n-r+1|1}$

Multiply these and all the P's cancel except $P_{n|r}$ and $P_{n-r+1|1}$ and the value of the latter is n-r+1

∴ $P_{n|r} = n(n-1)(n-2)\cdots(n-r+1)$ ■ [1]

Cor. 1. $P_{n|n} = n(n-1)\cdots 3\cdot 2\cdot 1 = \prod n_i$ [1-n]

Cor. 2. $P_{n|r} = n!/(n-r)!$

Cor. 3. The ways of arranging n letters in circular order is (n-1)! or (n-1)!/2 as clock-wise (cw) and counter-clock-wise (ccw) are or are not distinguished.

Thm. 1.19. When each of n letters can be repeated, $P_{n|r} = n^r$.

Thm. 1.20. Given n letters where groups α, β, γ, ... are the same letter $P_{n|n} = n!/\alpha!\beta!\gamma!\ldots$

Cor. 1. The ways of putting n things into r holes: α in first hole, β in second hole, ... is $n!/\alpha!\beta!\ldots$ (Note: Here the order of holes is fixed but the order of their contents is ignored.)

Circular Example

In how many ways can n distinct beads be made into a bracelet?
Since turning the bracelet over turns a cw arrangement into a ccw arrangement,
∴ the number is (n-1)!/2.

Combinations

Thm. 1.21. Given s sets of n_i things: $n_i \in \mathbf{N}$ [1-s], s things can be selected, one from each set in $n_1 n_2 \cdots n_s$ ways.

Thm. 1.22. The number of r-combs of n things ($C_{n|r}$) is

$$n(n-1)\cdots(n-r+1)/1\cdot 2\cdot 3\cdots r \qquad [1]$$

Proof

The proof is Sym. to the proof of $P_{n|r}$ but we have $rC_{n|r} = nC_{n|r-1}$ ∴ $C_{n|r} = n/r \cdot C_{n-1|r-1}$

Then enumerate the descending cases and multiply. But we already did that.

Here is a shorter proof: Every r-comb of n letters, if permuted in every way gives r! r-perms. And each r-perm occurs only once in this way.

∴ $r!C_{n|r} = P_{n|r}$ ∴ $C_{n|r} = P_{n|r}/r!$ ■

Cor. 1. Multiply numerator (num) and denom of $C_{n|r}$ by $(n-r)(n-r-1)\cdots3\cdot2\cdot1$

$\therefore\ C_{n|r} = n!/r!(n - r)!$

Cor. 2. $C_{n|r} = C_{n|n-r}$ and this follows from Cor. 1.

Cor. 3. $C_{n|r} = C_{n-1|r} + C_{n-1|r-1}$ and this is true of all f(n) $n\in\mathbf{N}$ of form [1] where n is unrestricted.

Cor. 4. $C_{n-1|s} + C_{n-2|s} + \cdots + C_{s|s} = C_{n|s+1}$

Cor. 5. $C_{p|s} + C_{p|s-1}C_{q|1} + C_{p|s-2}C_{q|2} + \cdots + C_{p|1}C_{q|s-1} + C_{q|s} = C_{p+q|s}$

Cor. 4 and 5 are propositions of series summation. More than that, we are establishing an arithmetic of combs. If you worked with small sets as concrete examples, you would realize the form of this arithmetic. Cor. 5. also takes the form:

$$(p(p-1)\cdots(p-s+1))/s! + (p(p-1)\cdots(p-s+2))/(s-1)! \cdot q/1 + \cdots$$
$$+ (p(p-1)\cdots(p-s+3))/(s-2)! \cdot (q(q-1))/2! +$$
$$p/1 \cdot (q(q-1)\cdots(q-s+2))/(s-1)! + (q(q-1)\cdots(q-s+1))/s! =$$
$$((p + q)(p + q - 1)\cdots(p + q -s + 1))/s! \qquad [2]$$

Cor. 6. Multiply both sides of [2] by s!, denote $p(p-1)\cdots(p-s+1)$ by p_s and we have **Vandermonde's Theorem**:

$$(p+q)_s = p_s + C_{s|1}p_{s-1}q_1 + C_{s|2}p_{s-2}q_2 + \cdots + q_s$$

Thm. 1.23. The r-combs of p+q letters, p alike, are

$$C_{q|r} + C_{q|r-1} + \cdots + C_{q|1} + 1$$
$$= q!(1/(r!(q-r)!) + 1/((r-1)!(q-r+1)!) + \cdots + 1/(1!(q-1)!) + 1/q!)$$

Cor. 1. The r-perms of same are:

$$q!r!(1/(r!(q-r)!) + 1/(r-1)!(q-r+1)!) + 1/(2!(r-2)!(q-r+2)!) + \cdots$$
$$+ 1/((r-1)!1!(q-1)!) + 1/(r!q!)) =$$
$$C_{q|r}r! + C_{q|r-1}\cdot r!/1! + C_{q|r-2}\cdot r!/2! + \cdots + C_{q|1}\cdot r!/(r-1)! + 1$$

Thm. 1.24. The r-combs of n letters, each letter repeated up to r times ($H_{n|r}$) is

$$\frac{(n + r - 1)!}{(n-1)!\, r!} = \frac{n(n+1)\cdots(n+r-1)}{r!}$$

Cor. 1. $H_{n|r} = C_{n+r-1|r}$

Cor. 2. $H_{n|r} = H_{n-1|r} + H_{n|r-1}$

Cor. 3. $H_{n|r} = H_{n-1|r} + H_{n-1|r-1} + H_{n-1|r-2} + \cdots + H_{n-1|1} + 1$

Cor. 4. The number of different r-ary products using n different letters is

$$(n(n+1)\cdots(n+r-1))/r!$$

and the number of terms in a complete ifn r° of n vars is

$$((n+1)(n+2)\cdots(n+r))/ r! = H_{n+1|r}$$

And once you go down this road, the H will haunt you, just like the C and P do. I have no idea what H stands for. We can restate the Binomial Theorem in terms of $C_{n|r}$:

$$(a + b)^n = a^n + C_{n|1}a^{n-1}b + C_{n|2}a^{n-2}b^2 + \cdots + C_{n|r}a^{n-r}b^r + \cdots + b^n$$

and this takes form:

$$(a + b)^n = \sum(n!/(\alpha_i!(n-\alpha_i)) \times a^{\alpha i}b^{n-\alpha i}$$

where α is always followed by a subscript and α_i takes all positive integers values such that $\sum\alpha_i = n$ and those last i's are both actually subscripts [1-n] and this is also

$$= \sum (n(n-1)\cdots(n-\alpha_i+1)/\alpha_i! \times a^{\alpha i}b^{n-\alpha i}$$

The **Multinomial Theorem** generalizes this

$$(a_1 + a_2 + \cdots + a_m)^n = \sum(n!/(\prod\alpha_i!) \times a_1^{\alpha 1}a_2^{\alpha 2}\cdots a^{\alpha m} \qquad [1]$$

Just think about it until you see the Binomial Theorem in this way. We can use this to find the coeff of x^r in the expansion of $(b_1 + b_2x + b_3x^2 + b_m^{m-1})^n$. The general term, using α as above and letting $\gamma = \alpha_2 + 2\alpha_3 + \cdots + (m-1)\alpha_m$, is:

$$n!/\prod\alpha_{i\,[1-m]} \times \sum b_i^{ai}{}_{[1-m]} \times x^\gamma$$

which looks ugly but is useable in practice. We want terms where γ = r and this coeff takes its form from [1] above. But let's use some examples to see what the heck we're talking about. Note that 0! = 1.

Examples

1) Req. coeff of a^3b^2 in $(a + b + c +d)^5$ is 5!/3!2!0!0! = 10
which is way easier than you expected.

2) Req. coeff of x^5 in $(1 + 2x + x^2)^4$
We need $\alpha_1 + \alpha_2 + \alpha_3 = 4$
$\alpha_2 + 2\alpha_3 = 5$
$\therefore \alpha_1 = \alpha_3 - 1 \quad \alpha_2 = 5 - 2\alpha_3$

α_1	α_2	α_3
0	3	1
1	1	2

$\therefore\ 4!/(0!3!1!) \cdot 1^0 2^3 1^1 + 4!/(1!1!2!) \cdot 1^1 2^1 1^2 = 56$
Note that $(1 + 2x + x^2)^4 = (1 + x)^8$ and coeff $x^5 = C_{8|5} = (8\cdot7\cdot6\cdot5\cdot4)/(1\cdot2\cdot3\cdot4\cdot5) = 56$

Both Chrystal and Todhunter go into this pretty deeply, in case you are hungry for more. I'm full now, thank you.

Thm. 1.25. If we have r sets of n_i [1-r] different letters, the number of ways of making combs by taking 1 to r letters at a time but never more than one from any set is

$$(n_1 + 1)(n_2 + 1)\cdots(n_r + 1) - 1$$

Proof

Consider the product
$$(1 + a_1 + b_1 + \cdots \text{ of } n_1 \text{ letters})$$
$$\times (1 + a_2 + b_2 + \cdots \text{ of } n_2 \text{ letters})$$
$$\times \cdots$$
$$\times (1 + a_r + b_r + \cdots \text{ of } n_r \text{ letters})$$

Here every comb of letters taken 1,2,3,... at a time occurs with the term 1 in addition. If we replace each letter by unity, each term in the product becomes unity and the sum will exceed the combs by 1. So sum above can be expressed:

$$\sum n_1 + \sum n_1 n_2 + \cdots + n_1 n_2 \cdots n_r \blacksquare$$

Let's find the number of ways n different letters can be put into r holes, every hole getting one or more letters, where we care about the order of holes but not the order of letters in them. Let D_r be the number. If we leave s holes empty, we have D_{r-s} in r - s holes. So the number of **distributions** (D) is $C_{r|s}D_{r-s}$. The total distributions when any or no holes are empty is r^n.

∴ $$D_r + C_{r|1}D_{r-1} + C_{r|2}D_{r-2} + \cdots + C_{r|r-1}D_1 = r^n$$

If we put r=2, r=3, ... here, we could calculate D_2, D_3,... and $D_1 = 1$. So we could get our answer. But we can do better. Sym. to above:

$$D_{r-1} + C_{r-1|1}D_{r-1} + C_{r-1|2}D_{r-2} + \cdots + C_{r-1|r-2}D_1 = (r-1)^n$$

Subtracting: $$C_{r|s} - C_{r-1|s-1} = C_{r-1|s}$$

∴ $$D_r + C_{r-1|1}D_{r-1} + C_{r|2}D_{r-2} + \cdots + C_{r-1|r-1}D_1 = r^n - (r-1)^n$$

If we continue this pattern of subtraction and derivation we get:

$$D_r = r^n - C_{r|1}(r-1)^n + C_{r|2}(r-2)^n - \cdots + (-1)C_{r|r-1}1^n$$

Cor. 1. If the order of holes is ignored, the number of distributions is $D_r/r!$ OR the number of partitions of n things into r lots, no empty lots, is $D_r/r!$ and just to let you know it's out there, the number of ways n things can be **deranged** so that none are in their original place is

$$n!(1 - 1/1! + 1/2! + \cdots + (-1)^n/n!)$$

which is known as **Whitworth's Subfactorial n**.

Combinatorics can be used to answer topological questions such as: There are n points in a plane, no 3 collinear with the exception of p which are all in the same line. Find the number of lines that can be made of these points and the number of triangles with these points as vertices.

Method

Take any pair of n-p points and we get (n-p)(n-p-1)/2! lines. Take any of the n-p points and one of the p = (n-p)p straight lines. And p in one line.

∴ (n-p)(n-p-1))/2! + p(n-p) + 1 lines

Or if none collinear (n(n-1))/2. With p collinear, we have one line instead of (p(p-1))/2

∴ (n(n-1))/2 - (p(p-1)/2 + 1 lines.

Next take any 3 of the n-p ∴ ((n-p)(n-p-1)(n-p-2))/3!
Then take 2 of the n-p points and 1 p ∴ (p(n-p)(n-p-1))/2!
Then take 1 of n-p and 2 of p ∴ (p(p-1)(n-p))/2! and sum for total.
OR if none are collinear (n(n-1)(n-2))/3!
But with p in one line we lose (p(p-1)(p-2))/3!
And the difference of these equals the above sum.

Substitutions

Combinatorics, beyond the basics, has little to interest me. But if you want to study Galois Theory, and I do, you will need to understand permutation groups and cyclic groups of, as it used to be called, Substitution Theory. Consider the letters abcde and their perms becda, bcade. Given becda, we can make bcade by changing a to e and c to a and e to c and represent this as:

(abcde:ebadc)becda = bcade

or more briefly as

(ace:eac)becda = bcade

where (ace:eac) is an operator and its effect is **substitution** (sub).
We can denote (ace:eac) as S, then

S(becda) = bcade

The substitution S^0 is the identity operator which leaves a perm unchanged or

S^0(becda) = becda

And we can do compound subs such as (abc:cab)≡S, (ae:ea)≡T then

ST(aebcd) = ecabd

which is S•T(aebcd) = S(eabcd) = ecabd. We denote SS as S^2:

S^2(aebcd) = S(ceabd) = becad

Note that S and T are not commutative: ST(perm) ≠ TS(perm) But if the subs are disjoint (no letters in common), the commute. And this condition is sufficient but not necessary as one sub can simply reverse another.

The perms of n letters are finite. So for ∀S, ∃μ: $S^\mu = S^0$. In other words, μ is the smallest number or **order** of S that returns the variable to its original state. Therefore, ∀ρ∈**N**, $S^\mu = S^{\rho\mu} = 1$ which is another way to write S^0.

Negative indices work this way: $S^{-q} = S^{\rho\mu-q}$: ρμ>q.
Then $S^qS^{-q} = S^qS^{\rho\mu-q} = S^{\rho\mu} = 1$ ∴ $(S^q)^{-1} = S^{-q}$ and we get inverse subs.
So if S≡(abcd:dabc) then S^{-1}≡(dabc:abcd) = (abcd:bcda) by simple rearrangement.

A set of subs such that the product of any of them is still in the set is a **group**. The number of these subs is the group's **order** and the number of letters or other objects operated on is the group's **degree**. Clearly, ∀A, S^n n∈**N** is a group and its order is μ as above.

A **cyclic sub** replaces each letter with its consequent: (abcde:bcdea). This can be simply denoted (abcde). A single letter cycle (a) is the identity cycle and a two letter cycle (ab) = (ba) is a **transposition** (trans). Clearly the order of a cycle is the degree of its group.

Thm. 1.26. Every sub is either cyclic of the product of a number of independent cycles.
Proof
A general approach: Consider ∀S = (abcdefgh:bfdcgaeh)
Here we have (abf), (cd), (eg), and (h)
∴ S = (abf)(cd)(eg)(h)
The cycles are independent and so commute and so their order of operation is indifferent. ■ Think about why this is an adequate proof.

Thm. 1.27. ∀cycle of n letters is a product of n-1 transpositions.
Proof
The cycle (abcd) = (ab)(bc)(cd) and this can be applied generally too. ■
Cor. 1. ∀sub = n-r transpositions where n is the number of letters displaced and r is the number of its proper cycles.
Proof
Using S = (abf)(cd)(eg)(h) above
= (ab)(bf)(cd)(eg)
where n = 7 and cycles = 3 ■

Prop. 1.1. The product of two trans which have two letters in common is the identity sub. You can prove this for yourself.

Prop. 1.2. In the product of two trans TT' with a common letter, T' can be placed first T'T if we replace the common letter with the other letter of T'.

(ab)(bc) = (abc:bca) = (bc)(ac)

Cor. 1. (ef)(af) = (ae)(ef)
Cor. 2. (ae)(af) = (af)(ef)

Prop. 1.3. If T, T' have no common factor, they commute.

Thm. 1.28. Decomposition of a sub into trans is not unique.
Proof
We can introduce a pair of factors (ab)(ab) and commutate them with the others by the above props. Doing this increases the trans by an even number.

Thm. 1.29. The number of trans equivalent to a given sub is always odd or always even. And you will see this theorem again.
Proof
This proof is also a method for reducing a product of trans to a standard form. Take the first letter "a" in its rightmost position. Move it left altering any trans with a by Prop. 1.2. If a duplicate trans occurs, remove both. Either all trans with "a" disappear or an even number (0,2,...) are removed and one remains on far left. Wash, rinse, repeat with remaining letters. Even or odd. ■ (Clearly, we can divide all subs into even or odd.)
Cor. 1. ∀S, if n = number of letters altered, r = number of cycles, 2s = arbitrary even integer, the number of factors in an equivalent product is n - r + 2s.
Cor. 2. Of all subs for n letters, odd subs = even subs.
Cor. 3. A cycle is even or odd as its letters are odd or even. (abc) = (ab)(bc)
Cor. 4. The product of any number of subs is even or odd as the number of odd factors is even or odd. Any power of an even sub or any even power of an odd sub is even.
Cor. 5. All the even subs of a set of n letters form a group of order n!/2

Symmetric fns of a set of n vars are unaltered in value by any sub (perm) of its vars. Alternating fns admit only even subs of their vars. This is as far as Chrystal takes substitution. And that was the best introduction to permutation groups and cyclic groups I have ever seen.

2. Integral Functions

The form of a polynomial function of one variable in general is:

$$f(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n \qquad [f]$$

When c_i [1-n] ∈ **Z**, we call this an **integral function** (ifn) and if $c_0 = 1$, we have a **regular integral function** (rifn). And if all coeffs ∈ **Q**, we call it a **rational function** (qfn). Be warned that "rational fn" is an overloaded term and can mean different things in different, especially older, texts. But we will always use rational function or qfn in this present sense. We will begin by looking at things even more generally than this dominant form.

Multinomial Factors

Consider (a + b)(c + d)(e + f) = what? You don't actually have to multiply this out to get the product. You can take one term from each factor, produce all partial products, and apply the law of signs:

$$\therefore \qquad = ace + acf + ade + adf + bce + bcf + bde + bdf$$

Let the factors have l, m, n, ... terms and the resulting product has l·m·n·⋯ terms. And above we had three factors with two terms ∴ 2·2·2 = 8 terms in the product. Before we do more examples here, there is something that we are rarely if ever told in algebra. And that is that the above polynomial can be considered a fn of any of its terms OR:

$$f(a) = (a + b)(c + d)(e + f) = ace + acf + ade + adf + bce + bcf + bde + bdf$$
$$= (ce + cf + de + df)a + bce + bcf + bde + bdf$$

Or it could be f(a,b) or f(a,c,e). You will see that this is a useful thing to know. Anyway, if we have (a + b)(a + b), we must have 2·2 terms. But the only possible terms are a·a, a·b, and b·b so one of them is repeated in $a^2 + 2ab + b^2$. Sym. in $(a + b)^3$ we must have eight terms with only four possible combinations. There are three ways to get a^2b (or ab^2) or $C_{3|2} = (3\cdot2/2\cdot1) = 3$: aab, aba, baa. And only one way to get a^3 or b^3. So we must have

$$a^3 + 3a^2b + 3ab^2 + b^3$$

With $(a + b + c)^2$ we must have 3·3 = 9 terms. Only one way to get any α^2 and only two ways to get any $\alpha\beta$:

$$(a+ b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

which gives us nine terms, three terms combined of two. What about $(a + b + c)^3$? The point here is that by examining the form, we don't have to multiply anything. There is one way to get any α^3; three to get any $\alpha^2\beta$; and six to get abc. If you will work until you can **see** this, you can see these three numbers --1,3,6 -- and write:

$$a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3a^2c + 3ac^2 + 6abc$$

There is an old use of the capital sigma "∑" where it means "sum of combinations of this form". There are fields in algebra where it is still used and, of course, you can use it too if you find it helpful. Using it, we have:

$$(a + b)^3 = \sum a^2 + 3\sum a^2b$$
$$(a + b + c)^3 = \sum a^3 + 3\sum a^2b + 6abc$$
$$(b + c)(c + a)(a + b) = \sum a^2b + 2abc$$

Using what you have just learned, check that last one to make sure it's correct. (I don't actually catch all my typos.) And then calculate $(b - c)(c - a)(a - b)$ which does not lend itself to sigma notation. From all this, we can see that the form here dictates the following:

$$(a + b + c + d + \cdots)^2 = \sum a^2 + 2\sum ab$$
$$(a + b + c + d + \cdots)^3 = \sum a^3 + 3\sum a^2b + 6\sum abc$$

There is also a capital pi "∏" notation for products of combinations. For a function in a,b,c, $\prod a^2b = a^2b \times ab^2 \times a^2c \times ac^2 \times b^2c \times bc^2$. So we could say that $(b + c)(c + a)(a + b)$ is $\prod(b + c) = \sum a^2b + 2abc$. Another way to look at expanding a known identity, like the ones above, is to use our Principle of Substitution. I should add that an **identity** is simply a tautological eqn. The LHS equals the RHS no matter what you feed it. We know that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

So we can let $b = b+c$ and then

$$(a + b + c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$$

and algebrate. And in these expansions, don't forget to look for your old friend:

$$(a + b + c - d)(a - b + c + d)$$
$$= ((a + c) + (b - d))((a + c) - (b - d))$$
$$= (a + c)^2 - (b - d)^2$$

Thm. 2.1. If all terms in all factors are a single letter without coeff and are all positive then sum of coeff in product equals product of numbers of factors.
Example
$(a + b)^3 \Rightarrow 1 + 3 + 3 + 1 = 2 \times 2 \times 2$

Integral Functions of One Variable

A quick review from DME on these integral functions (ifn). The sum, difference, and product of ifns are ifns. If ifn A divides ifn B without remainder, the quotient is an ifn. The highest/lowest terms in an ifn are the products of the highest/lowest terms in its factors. And the degree of the product of ifn is the sum of the degrees of its factors. In standard form [ƒ], a fn of degree n has n+1 terms, any of which may have coeff of zero.

A few remarks to move things a bit further: If all the factors are in the form $(x - \alpha_i)$ for roots α_i [1 - n], the terms of the product begin positive and alternate sign:

$$\begin{aligned}
&(x - a)(x - 2a)(x - 3a)(x - 4a) \\
&\quad = x^4 + (1 + 2 + 3 + 4)ax^3 - (1\cdot2 + 1\cdot3 + 1\cdot4 + 2\cdot3 + 2\cdot4 + 3\cdot4)a^2x^2 \\
&\qquad\quad + (1\cdot2\cdot3 + 1\cdot3\cdot4 + 1\cdot2\cdot4 + 1\cdot3\cdot4)a^3x - (1\cdot2\cdot3\cdot4)a^4 \\
&\quad = x^4 - 10x^3a + 35x^2a^2 - 50xa^3 + 24a^4
\end{aligned}$$

This is both an example of the alternation of sign and of the relation of coeffs to roots. If all the roots are in form $(x - \alpha)$, then the first coeff is unity. Using our other notation of N_r for choosing r things from n things, the things are the roots, the degree of the fn is N, and the coeffs of the terms following the first are constructed by choosing N_i where i is the term (2d, 3d, 4th, ...), take the product of each combination, and sum them up. And the example shows that this form remains the same whether the roots are 1,2,3,4 or 1a,2a,3a,4a:

$$\begin{aligned}
4_0 &= 1 \;\therefore\; c_0 = 1 \\
4_1 &= 4 \;\therefore\; c_1 = a + 2a + 3a + 4a \\
4_2 &= 6 \;\therefore\; c_2 = a\cdot2a + a\cdot3a + a\cdot4a + 2a\cdot3a + 2a\cdot4a + 3a\cdot4a \\
4_3 &= 4 \;\therefore\; c_3 = a\cdot2a\cdot3a + a\cdot3a\cdot4a + a\cdot2a\cdot4a + 2a\cdot3a\cdot4a \\
4_4 &= 1 \;\therefore\; c_4 = a\cdot2a\cdot3a\cdot4a
\end{aligned}$$

I'm sure that if you tried, you could prove this generally for n roots by induction. We had these ideas in DME and I guarantee you that they will not go away.

$n \in \mathbf{N}$	$x^n - y^n$	divby	$x - y$	remainder: all terms +
n odd	$x^n + y^n$	divby	$x + y$	remainder: terms alternate + - ends +
n even	$x^n - y^n$	divby	$x + y$	remainder: terms alternate + - ends -

I bring these up to show you how with ones, all things are easier. Let's multiply x - y by the all pos-term remainder to prove the first one:

$$\begin{array}{lllllll|l}
1 + 1 + 1 + \cdots + 1 + 1 & & & & & & & = \text{remainder} \\
\underline{1 - 1 \qquad\qquad\qquad\quad .} & & & & & & & = x - y \\
1 & 1 & 1 & \cdots & 1 & 1 & & \\
\underline{\quad} & -1 & -1 & \cdots & -1 & -1 & -1 & \\
1 & & & & & & -1 & = x^n - y^n
\end{array}$$

If that doesn't seem right, use a small n for a test. Another way ones make things easy is this: consider $x^3 + x^2 + x + 1$. We can square it with detached coeffs:

1	1	1	1							
	1	1	1	1						
		1	1	1						
			1	1	1	1				
1	2	3	4	3	2	1				and then cube it:
	1	2	3	4	3	2	1			
		1	2	3	4	3	2	1		
			1	2	3	4	3	2	1	
1	3	6	10	12	12	10	6	3	1	which is:

$$x^9 + 3x^8 + 6x^7 + 10x^6 + 12x^5 + 12x^4 + 10x^3 + 6x^2 + 3x + 1$$

Homogeneity

An integral function of any number of variables is **homogeneous** (homog) if the degree of every term in the same. Think of the binomial expansion of $(x + y)^4$. The degree of every term is four: x^4, $4x^3y$, etc. Only variables have degree. If in $3a^2x$, a is a constant, degree is one. If a is a var, degree is 3. If the number of terms is N and the degree is n then:

$$N = 1/2 \cdot (n+1)(n + 2)$$

which is the sum of the first n+1 natural numbers.

Thm. 2.2. If in an homog fn, n°, each var is multiplied by p, the result is the same as fn multiplied by p^n.

This could be used as a definition of an homogeneous function. Expand $(pa + pb)^2$ to see what this means.

Thm. 2.3. ∀homog fn, 1°, for x,y,z,... sub $\lambda x_1+\mu x_2$, $\lambda y_1+\mu y_2$, $\lambda z_1+\mu z_2$,... then the result is the same as subbing x_1,y_1,z_1,... and x_2,y_2,z_2,... respectively for x,y,z,... after multiplying sums by λ and μ respectively.

Example

$Ax + By + Cz \Rightarrow A(\lambda x_1+\mu x_2) + B(\lambda y_1+\mu y_2) + C(\lambda z_1+\mu z_2)$

$= \lambda(Ax_1 + By_1 + Cz_1) + \mu(Ax_2 + By_2 + Cz_2)$

This shows up again in analytical geometry and linear algebra.

Law of Homogeneity

The product of two homog ifns, degrees m°, n°, is an homog ifn (m+n)°

Symmetry

1) A fn is **symmetric wrt ∀2 vars** if the two vars can be exchanged without changing the value of the fn.

(a + b)(a + b + c) is symmetric wrt a,b but not a,c or b,c.

This is easier to see in the factors than in the product.

2) A fn is **symmetric wrt all vars** when the interchange of any two vars does not affect the value of the fn.

$x^2y + y^2z + z^2x$ is not symmetric at all.

Exchanging any two vars changes the value.

3) A fn is **collaterally symmetric in two sets of vars** {x_i [1-n]} and {a_i [1-n]} when the simultaneous exchange of two from each set leaves the value unchanged.

$a^2x + b^2y + c^2z$ { a,b,c } { x,y,z }

Exchanging x⇔y and a⇔b: $b^2y + a^2x + c^2z$ ∴ collaterally symmetric.

We will often use "sym" to denote "symmetric" or "symmetrical" in the above sense and keep "Sym." to denote "by symmetric proof."

Rule of Symmetry
The sum, product, and quotient of two symmetric fns is a symmetric fn. To make a fn symmetric:

$Ax + By$	$A = B$	$Ax + Ay$
$Ax^2 + Bxy + Cy^2$	$A = C$	$Ax^2 + Bxy + Ay^2$
$Ax^3 + Bx^2y + Cxy^2 + Dy^2$	A,B = D,C	$Ax^3 + Bx^2y + Bxy^2 + Ay^2$

Symmetry gives us even more leverage to produce products of multinomial factors based on the form of number rather than by calculation:

$(a + b + c)(a^2+ b^2+ c^2 - bc - ca - ab)$
two symmetric terms ∴ product symmetric:
a^3 coeff 1 ∴ b^3,c^3 sym.
no b^2c ∴ no c^2a, a^2c, bc^2, ab^2, ba^2
clearly, -3abc
∴ product: $a^3 + b^3 + c^3 - 3abc$

With ifns, there is a "Principle of Indeterminate Coefficients", which is one of the worst name choices in mathematics. It says that because the coeffs of an ifn are independent (ind.) of x, once they are in any way determined, they are fixed. Which is to say that the coefficients of a fn are determined by the form of the factors and not by the variables and that this determination is absolute.

Examples

1) $(x + y)^2 = (x + y)(x + y) = Ax^2 + Bxy + Ay^2$ [1]
This must be true of any x and any y.
Take $(1 + 0)^2 = A\cdot1^2 + B\cdot1\cdot0 + A\cdot0^2 \therefore A = 1$
$\therefore Ax^2 + Bxy + Ay^2 = x^2 + Bxy + y^2$
Take $(1 + -1)^2 = 1 + B\cdot1\cdot-1 + 1 \therefore B = 2 \therefore x^2 + 2xy + y^2$
This is also solvable as a system of 1° eqns.
Using [1] with points (2,3) and (1,4):
$25 = 13A + 6B$
$25 = 17A + 4B$
$\therefore A = 1 \ B = 2$

2) $(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)$ [2]
By form this must equal $A\sum x^3 + B\sum x^2y + Cxyz$ [3]
Let $x = 1$, $y,z = 0 \therefore A = 1$
Let $x,y = 1$, $z = 0$, $A = 1 \therefore$ [2] = [3] $\therefore 2\cdot1 = 2 + B\cdot2 \therefore 2B = 0 \therefore B = 0$
(If you can't see why we have $B\cdot2$, expand $B\sum x^2y$.)
Then let $x,y,z = 1$, $A = 1$, $B = 0 \therefore 3\cdot0 = 3 + C \therefore C = -3$
$\therefore$ product is $x^3 + y^3 + z^3 - 3xyz$

Algebraic Forms

I'm about to give you Chrystal's huge list of identities which I prefer to think of as common algebraic forms of number. Let me tell you a story first. If you wanted to be a cab driver in London, you would go to the School of Knowledge. And there, for ten years, you would study the map of London until you knew every one of its twenty-five thousand streets and a great deal about what is on those streets. I am not making this up. The tests have questions like: "Your customer wants to go from his home on [obscure street] to his mother's house on [obscure street opposite side of London]. What is the shortest route?" I learned this on a radio program and hearing someone reel off the route across London was amazing. When asked if he had always had a good memory, he said, no.

You can do more with your mind than you think you can. All of the following forms are common enough to bear memorizing. Think about doing just that. It will not take you ten years. It might not even take you a weekend. And once they are in your head, they are there to consider whenever you like.

What follows is the table of identities from Chrystal's *Algebra - An Elementary Text Book*.

I.

$(x + a)(x + b) = x^2 + (a + b)x + ab$

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (bc + ca + ab)x + abc$

and generally

$(x + a_1)(x + a_2) \cdots (x + a_n) = x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots + P_{n-1}x + P_n$

[where P_i = sum of "n choose i of the a's"]

$(x \pm y)^2 = x^2 \pm 2xy + y^2$

$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$ &c.;

and so on, the numerical coefficients being taken from the following table of binomial coefficients:

II.

Power	Coefficients												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

III.

$(x \pm y)^2 \mp 4xy = (x \mp y)^2$

IV.

$(x + y)(x - y) = x^2 - y^2$

$(x \pm y)(x^2 \mp xy + y^2) = x^3 \pm y^3$

and generally

$(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = x^n - y^n$

$(x + y)(x^{n-1} - x^{n-2}y + \cdots \mp xy^{n-2} \pm y^{n-1}) = x^n \pm y^n$

upper or lower sign accordingly as n is odd or even.

V.

$(x^2 + y^2)(x'^2 + y'^2) = (xx' \mp yy')^2 + (xy' \pm x'y)^2$

$(x^2 - y^2)(x'^2 - y'^2) = (xx' \pm yy')^2 - (xy' \pm x'y)^2$

$(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) = (xx' + yy' + zz')^2 + (yz' - yz')^2 + (zx' - z'x)^2 + (xy' - x'y)^2$

$(x^2 + y^2 + z^2 + u^2)(x'^2 + y'^2 + z'^2 + u'^2) = (xx' + yy' + zz' + uu')^2$
$+ (xy' - yx' + zu' - uz')^2$
$+ (xz' - yu' - zx' + uy')^2$
$+ (xu' + yz' - zy' - ux')^2$

VI.

$(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4$

VII.
$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
and generally
$(a_1 + a_2 + \cdots + a_n)^2$ = sum of the squares of $a_1, a_2, ..., a_n$
+ twice sum of all partial products two and two.

VIII.
$(a + b + c)^3 = a^3 + b^3 + c^3 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 3a^2b + 3ab^2 + 6abc$
$= a^3 + b^3 + c^3 + 3bc(b + c) + 3ca(c + a) + 3ab(a + b) + 6abc$

IX.
$(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3abc$

X.
$(b - c)(c - a)(a - b) = -a^2(b - c) - b^2(c - a) - c^2(a - b)$
$= -bc(b - c) - ca(c - a) - ab(a - b)$
$= bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b$

XI.
$(b + c)(c + a)(a + b) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$
$= bc(b + c) + ca(c + a) = ab(a + b) + 2abc$
$= bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc$

XII.
$(a + b + c)(a^2 + b^2 + c^2) = bc(b + c) + ca(c + a) + ab(a + b) + a^3 + b^3 + c^3$

XIII.
$(a + b + c)(bc + ca + ab) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc$

XIV.
$(b + c - a)(c + a - b)(a + b - c) =$
$a^2(b + c) + b^2(c + a) + c^2(a + b) - a^2 - b^2 - c^2 - 2abc$

XV.
$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 2b^2c^2 + 2a^2c^2 + 2a^2b^2 - a^4 - b^4 - c^4$

XVI.
$(b - c) + (c - a) + (a - b) = 0$
$a(b - c) + b(c - a) + c(a - b) = 0$
$(b + c)(b - c) + (c + a)(c - a) + (a + b)(a - b) = 0$

Polynomial Division

If we have any ifns A,D in one var and ifn Q: D·Q = A then Q≡quotient, A≡dividend, and D≡divisor. If A,D coeff∈**Z** then Q coeff∈**Q**. When Q can be transformed into coeff∈**Z**, then A divby D else Q is fractional OR:

Q ifn ⇒ Q° = A° - D°
A° < D° ⇒ Q fractional

Thm. 2.4. A,D,Q,R ifn: A = QD + R ⇒ A and R divby D
Proof
A = QD + R ∴ A/D = Q + R/D ∴ R/D = A/D - Q ■

Because ifns are "as integers" in Euclid's Algorithm, these theorems follow from the proofs in DME for arithmetic division.

Thm. 2.5. A_m,D_n ifns m°,n° then we can transform A_m/D_n: $A_m/D_n = P_{m-n} + R/D_n$ where P_{m-n} ifn (m-n)°. If R ≠ 0, R ifn max (n-1)° Which is to say, *Dividing A_m by D_n, we get a quotient P_{m-n} plus a remainder in exactly the same form as in integral arithmetic.*

Thm. 2.6. The quotient A/D of two ifn takes the unique form: P + R/D where P,R ifns and R° < D°

We know that A divby D ⇔ R = 0. So if the divisor is n° then R is (n-1)° with n coeffs ∴ n conditions of divisibility.

We covered polynomial division in DME and mentioned that polynomials of more than one variable could also be divided and that "prime" factors could be found. But this would require special ordering of the polynomials involved. Consider:

$$a^4 - 3a^3 + 6a^2b^2 - 3ab^2 + b^4 \div a^2 - ab + b^2$$

Here we must choose a or b as var and then order appropriately. If we take a as var the order here is correct

```
1  -3   6  -3   1 |1  -1   1
        etc.      |1  -2   3  =  a² - 2ab + 3b²
              2  -2 = 2ab³ - 2b⁴
```

If we choose b as var and order both appropriately then $Q = b^2 - 2ba + 3a^2$ and $R = 2ba^3 - 2a^4$ which shows that these fns are symmetric. In DME, we proved the Remainder Theorem and used Synthetic Division to demonstrate its use. This gives us one way to factor ifns.

Thm. 2.7. If a_i [1-r] are roots of ifn f(x), n°: r < n ⇒
$f(x) = (x - a_1)(x - a_2)(x - a_3)\cdots(x - a_r)\cdot\varphi x$ where φ is ifn (n-r)°
Cor. 1. ifn f(x) divby $(x - a_i)$ [1-r]: $(x - a_i)$ 1° and a_i distinct ⇔ f(x) divby $\prod(x - a_i)$
Cor. 2. a_i [1-n] roots of f(x), n° ⇒ f(x) decomposable into n factors $(x - a_i)$
Cor. 3. ifn f(x) vanishes for more than n factors ⇒ each of its coeffs must vanish
Example $(x+1)(x-1) - x^2 + 1$ has roots 1,2,3 but it resolves to $0x^2 + 0$
Cor. 4. If two ifn m°>n° be equal for more than m different values of x, *a fortiori* if they be equal for ∀x (identically equal) ⇒ coeffs of like powers must be equal.

Cor. 4 is not trivial in the way our DME theorem about equating coeffs in infinite series was not trivial. In fact, Cor. 4 is the Principle of Indeterminate Coefficients redux. And it implies that there is a unique transformation of any f(x) into an ifn. Let's show some uses of the Remainder Theorem:

Examples

1) Determine k: $x^3 + 6x^2 + 4x + k$ is divby (x + 2)

Using Synthetic Division from DME:

1 6 4 k | -2
-2 -8 8 |
1 4 -4

Here we have remainder 8 + k

For divby, need 8 + k = 0 ∴ k = -8

2) Is $3x^3 - 2x^2 - 7x - 2$ divby (x + 1)(x - 2)?

f(-1) = -3 - 2 + 7 - 2 = 0 divby (x + 1)

f(2) = 24 - 8 - 14 - 2 = 0 divby (x - 2)

So by Remainder Theorem f is divby both factors.

∴ 3d factor 1° = ax + b ∴ a = 3

Constant term = -2 ∴ -1·2·b = -2 ∴ b = 1 ∴ remaining factor is (3x + 1)

3) n∈**N**

when	divided by	remainder	meaning
$x^n - a^n$	x - a	$a^n - a^n$	0 always
$x^n - a^n$	x + a	$(-a)^n - a^n$	0 (n even), $-2a^n$ (n odd)
$x^n + a^n$	x - a	$a^n + a^n$	$2a^n$ always
$x^n + a^n$	x + a	$(-a)^n + a^n$	0 (n odd), $2a^n$ (n even)

4) Show $a^3(b - c) + b^3(c - a) + c^3(a - b) = -(a + b + c)(b - c)(c - a)(a - b)$

Consider LHS as fn of a ∴ vanishes when a = b ∨ c

∴ divby (a - b) and (a - c)

LHS as f(b) vanishes if b = c ∴ divby (b - c)

∴ LHS = Q(a - b)(a - c)(b - c)

LHS as f(a,b,c) is 4° ∴ Q(a,b,c) must be 1° as factors wrt ∀var are 3°

∴ Q = (la + mb + nc)

∴ RHS = (la + mb + nc)(a - b)(a - c)(b - c) = -(la + mb + nc)(a - b)(c - a)(b - c)

Both fns are symmetric∴ for l determine coeff a^3b both sides

∴ l = 1 and by sym. m,n = 1 ∴ LHS = RHS

Factoring Polynomials

Consider

$$\frac{x^3+3x^2+3x+2}{3x^3+8x^2+5x+2}$$

If we let x = 10, this becomes

$$\frac{1332}{3852} = \frac{2^2 \cdot 3^2 \cdot 37}{2^2 \cdot 3^2 \cdot 41}$$

∴ gcm(1332,3852) = 36

If we express this in terms of x = 10, 36 becomes 3x + 6.
Dividing our two ifn by (3x + 6), we have

$$\frac{⅓x^2 + ⅓x + ⅓}{x^2 + ⅔x + ⅓}$$

But (3x + 6) is actually 3(x + 2) and if we use (x + 2) as our factor we have:

$$\frac{x^2 + x + 1}{3x^2 + 2x + 1}$$

What we have done is realized that the factors of a polynomial like this are the factors no matter what x is. So they are the factors when x=10 and therefore are the factors of f(10). Do we need to prove this? If you think about it, we don't. Recalling how ridiculous some of the greatest minds of the 20th century looked when they tried to prove arithmetic, I think we can skip that. This method is only arithmetic. And at first, the method strikes you as simple and powerful. Well, it is and it isn't. Or it isn't and it is, depending on how you look at it.

I call this **Niemand's Method** after Lewis Carroll's friend Herr Doktor Niemand in his *Euclid and His Modern Rivals*. It's a kind of joke. But Niemand's Method is not a joke. It is more of a heuristic than a rigorous method. But it delivers. Let me show you. We call the value of f(10) the **nval**. Here are some basic examples.

$x^2 + 15x + 36$ nval(286) = 2·11·13 (x + 3)(x + 12) ∴ 13·22	$x^2 + 9x - 36$ nval(154) = 2·7·11 (x - 3)(x + 12) ∴ 7·22
$x^2 - 15x + 36$ nval(-14) = -1·2·7 (x - 3)(x - 12) ∴ 7 · -2	$x^2 - 9x - 36$ nval(-26) = -1·2·13 (x + 3)(x - 12) ∴ 13 · -2
$2x^2 + 8x + 6$ nval(286) = 2·11·13 (2x + 6)(x + 1) ∴ 26·11	$2x^2 + 8x + 5$ nval(285) = 3·5·19 (2x + a)(x + b) integral ab must be 1·5 ∴ no integer soln for factors 3·5·19 (soln ≡ solution)

$2x^2 + 8x + 6 = 2(x^2 + 4x + 3)$ nval(143) = 11·13 (x + 3)(x + 1) ∴ 13·11 ∴ 2(x + 3)(x + 1)	$x^2 + x + 3$ nval(113)≡prime ∴ irreducible (complex factors)

How useful is Niemand's Method? For ifns with positive coefficients, it is the supreme method of factorization. You can see that the nval of a quadratic can have more than two factors and this excess is even more true for fns of higher degrees. But for positive coeffs, this is still the quickest method.

Let just one coeff be negative and it's a whole 'nother story. We know the relation of coeffs to roots. If a cubic has roots a,b,c, the first coeff is 1, the second a+b+c, the third is ab+bc+ca, and the fourth is abc. If a is negative we have b+c-a, bc-ab-ca, and -abc. But that doesn't make any but the last coeff negative necessarily. And if the last coeff is negative, we can only know that an odd number of roots are negative. Consider:

$$3x^3 - 4x^2 - 17x + 6$$

The nval is $2436 = 2^2 \cdot 3 \cdot 7 \cdot 29$. We can see there are negative roots and that there must be two of them due to the last coeff. And our factors must be (3x+a)(x+b)(x+c). If we combine the 7 with the 2, 3 or 4, the factor is (x+4), (3x-9), or (3x-2) with only the last being possible. The 7 alone gives (x - 3). We can use synthetic division to discard (3x - 2) and accept (x - 3) ∴ (3x+a)(x+b)(x-3). The 29 is clearly (3x-1), root ⅓, and is easily validated. Which leaves $2^2 \cdot 3 = 12 \Rightarrow (x+2)$ and we're done.

The nval is consistent in polynomial division. Let's look at fns with nvals of 144:

> $x^2 + 4x + 4$ factors into $(x + 2)^2$ or 12·12 = 144.
> $2x^2 - 6x + 4$ factors into (x + 2)(2x - 10) r 24 and 12·10 + 24 = 144.
> But $2x^2 - 6x + 4$ also factors evenly into (x - 2)(2x - 2) or 8·18 = 144.
> $3x^2 - 16x + 4$ factors into (x - 2)(3x - 10) r -16 and 8·20 - 16 = 144.

One thing this points to is that this method is more useful with regular ifn where $c_0 = 1$.

We can use the nval backwards to investigate the possibility of factors. With $3x^2 - 16x + 4$ we would need (3x - a)(x - b) just from the form of things and -a · -b = 4. The nval is 144. So from the form of the factors, we can ask ourselves, using "?" as a digit's placeholder, can there be a 2? · ? = $144 = 2^4 \cdot 3^2$. The only possibility for a twenty-something is 3·8 = 24 = (3x - 6) and the -6 scotches our required 4.

$x^3 + 12x^2 - 32x - 256$ has an nval of $1624 = 2^3 \cdot 7 \cdot 29$. The 12, 32, and 256 encourage us to believe f(x) is divby (x + 4) and synthetic division shows us this is so and with a quotient of $x^2 + 8x - 64$. Our (x + 4) has a root of -4 and an nval of 14. 1624÷14 = 116 and this is the nval of our quotient with factors of $2^2 \cdot 29$. The 64 and the 29 tell us there are no more integral factors and that the product of the surd roots is -64. The roots are $(-4 \pm 4\sqrt{5})/4$.

If you examine algebra texts, the methods of factoring polynomials are *ad hoc* attempts using the Remainder Theorem and the final coeff and the canonical extraction of roots for quadratics, cubics, and quartics. And for quintics and beyond, they throw up their hands in despair. Niemand's Method of factoring is clearly superior to the *ad hoc* method. It doesn't pretend to replace the canonical methods as it doesn't deliver on real or imaginary surds. What about functions of higher degrees?

Eisenstein's Criterion says that for an ifn with coeff∈**Z**, if ∃prime p: p divides any c_i but the last one and does not divide the last one (c_n), then f(x) is irreducible in **Q[x]** (polynomials with rational coeff). So $x^7 + 6x^5 - 15x^4 + 3x^2 - 9x + 12$ is irreducible using p=5. The nval here is 10450222 = 2·53·311·317. We need seven roots and ±1 do not work. So we have three pair of real or imaginary surds, together with an 8 from (x - 8) for the 2 and then $8 \cdot a_1 \cdot a_2 \cdot b_1 \cdot b_2 \cdot c_1 \cdot c_2 = 12$, right? No, f(8)≠0. As soon as surds enter, Niemand is no help. In $x^2 + x + 3$, the nval is 113 and prime. Yet there is no direct relation apparent between the 113 and the easily derived roots. But we can see even more clearly than Sergei here (I assume this Eisenstein guy is the film director) what we are dealing with. If we can ever relate the prime factors of the nval to the roots **in any way**, we would have even more to work with, if not the roots themselves. And if we are only talking about ifn of higher degrees with positive coeffs, Niemand rocks.

As we continue, I will point out where Niemand's Method is useful in our work. Perhaps someday it will amount to more than an often useful heuristic. But, hey, these are Niemand's ideas. I'm just his amanuensis in this world, he being already dead when he appeared as a ghost to Lewis Carroll.

Let's look at more tools for factoring ifns. Form can reveal factors:

$x^2 - y^2 \Rightarrow$ factors of (x + y) and (x - y)
$x^n \pm y^n \Rightarrow$ factors of (x + y) and/or (x - y)
$x^3 + y^3 + z^3 - 3xyz \Rightarrow$ factor of (x + y + z)

But form is only a help if you have the forms in your head or written out next to you where you can refer to them.

Examples

1) $x^2 - 12x + 32$ Assume ∃(x - a)(x - b)
$\therefore x^2 - 12x + 32 = x^2 - (a + b)x + ab$
$\therefore a + b = 12 \wedge ab = 32$
$\therefore a = 4 \wedge b = 8$

2) $x^3 - 2x^2 - 23x + 60 = (x - a)(x - b)(x - c) \wedge -abc = 60$

```
1  -2  -23  60 | 3          ( factor (x - 3) )
    3    3 -60 |
1   1  -20   0
```

$\therefore (x - 3)(x^2 + x - 20) = (x - 3)(x - 4)(x + 5)$

3) $6x^2 - 19x + 15 = (ax + b)(cx + d)$
$\therefore ac = 6 \wedge bd = 15$
$\therefore (2x - 3)(3x - 5)$

You can see that these tools complement the *ad hoc* method of factoring. At every step you are judging possibilities and validating them. Here's a higher level tool:

If ifn P can be arranged as a sum of terms where Q is a factor of each, Q is a factor of P. If P can be arranged into such a form and also an additional group where Q is not a factor then Q is not a factor of P.

Examples

1) $x^3 - 2x^2 - 23x + 60$
$= x^2(x - 2) - 23(x - 2) + 14 \therefore (x - 2)$!factor
$= x^2(x - 3) + x^2 - 23x + 60$
$= x^2(x - 3) + x(x - 3) - 20x + 60$
$= x^2(x - 3) + x(x - 3) - 20(x - 3) \therefore (x - 3)$ factor

And as a side note, we know from other ideas that this shows that $x^3 - 2x^2 - 23x + 60$ is the fn $x^2 + x - 20$ in terms of $(x - 3)$. And further, that this is an analog of changing bases with integers.

2) $px^2 + (1 + pq)xy + qy^2$
$= px^2 + xy + pqxy + qy^2$
$= x(px + y) + qy(px + y) \therefore (px + y)$ factor
$= (x + qy)(px + y)$

Let's go more deeply into the idea of quadratic factors and roots.

To make $x^2 + px + q$ a perfect square (perfect^2):

$$x^2 + px + q + \alpha = (x + \beta)^2 = x^2 + 2\beta x + \beta^2$$
$$\therefore p = 2\beta \wedge a = \beta^2 - q = (p/2)^2 - q$$
$$\therefore x^2 + px + q + (p/2)^2 - q = (x + \beta)^2$$

For the general form of $ax^2 + bx + c$ this becomes:

$$ax^2 + bx + c + (b^2 - 4ac)/4a = a(x + b/2a)^2$$
$$\therefore ax^2 + bx + c = a\left((x + b/2a)^2 - \left((b^2 - 4ac)/4a^2 \right) \right)$$

Let this take the form of $a((x + l)^2 - m^2)$
where $l = b/2a$ and $m = \sqrt{((b^2 - 4ac)/4a^2)}$
$\therefore$ factors are $a((x + l) + m)((x + l) - m)$
and now we have the form $(a + b)(a - b) = a^2 - b^2$

Examples

1) $x^2 + 2x - 1 = x^2 + 2x + 1 - 2$
$= (x + 1)^2 - (\sqrt{2})^2$
$= (x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

2) $x^2 + 2x + 5 = x^2 + 2x + 1 + 4$
$= (x + 1)^2 - (2i)^2$
$= (x + 1 + 2i)(x + 1 - 2i)$

3) $x^2 + 2x + 3 = x^2 + 2x + 1 + 2$
$= (x + 1)^2 - (i\sqrt{2})^2$
$= (x + 1 + i\sqrt{2})(x + 1 - i\sqrt{2})$

4) $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2$
$= (x^2 + y^2)^2 - (\sqrt{2}\cdot xy)^2$
$= (x^2 + y^2 + \sqrt{2}xy)(x^2 + y^2 - \sqrt{2}xy)$
And then
$x^2+\sqrt{2}xy + y^2 = (x + \sqrt{2}/2\cdot y) - 1/2\cdot y^2$
and so on.

It follows that in $ax^2 + bx + c$ [1]
that we can have a,b,c ∈ **Q** ∴ l = b/2a ∈ **Q** and

1. if $b^2 - 4ac$ is a positive square of a rational number then m is rational and [1] is (x + l + m)(x + l - m) as the product has two linear factors with coeff ∈ **Q**
2. if $b^2 - 4ac$ is positive but not a square of some q∈**Q** then m ∈ **R-Q** and then the coeffs must be in **R-Q**
3. if $b^2 - 4ac$ is negative then m∈**C-R** (that would be imaginary) and coeff ∈ **C**
4. if $b^2 - 4ac$ vanishes then m = 0 and [1] is $a(x + l)^2$, two factors real and identical

You can see how, in all of these quadratic ideas, Chrystal has taken a different approach than De Morgan did. Todhunter takes a third approach. And every real mathematician has his own understanding to share of even basic ideas like these. If you find yourself studying a book where the author **does not** have a distinct and interesting viewpoint, chuck it and go find a better book. Lots of people have **compiled** mathematics texts but have nothing of themselves to offer. You want an individual thinker to share his or her ideas with you. Mathematics is the product of **individual** thought.

More generally, for f(x), n° with coeff ∈ **N** there are an even number of roots ∈ **C** which we will see are in conjugate pairs. The function will then have n factors: p roots∈**R**, 2q∈**C**: p + 2q = n.

When the number of vars is greater than one, factorization is not generally algebraically solvable. An exception is homogeneous functions of two vars. In DME, we looked at the general second degree equation of two variables and showed how is was factorable. Let's look at this a little more deeply from Chrystal's point of view.

General form:

$$ax^2 + 2hxy + by^2 + 2qx + 2fy + c$$
$$= a(x^2 + 2 ((hy + g)/a)x + (by^2 + 2fy + c)/a)$$
$$= a(x^2 + 2Px + Q)$$
$$= a((x + P)^2 - (P^2 - Q))$$
$$= a(x + P + \sqrt{(P^2 - Q)})(x + P - \sqrt{(P^2 - Q)})$$

These are rational factors if $\sqrt{(P^2 - Q)}$ is a rational function in y.
∴ $P^2 - Q$ must be a perfect^2 wrt y

$$P^2 - Q = ((hy + g)^2 - a(by^2 + 2fy + c))/a^2$$
$$= ((h^2 - ab)y^2) + 2(gh - af)y + (g^3 - ac))/a^2$$

And this is a perfect^2 of

$$4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0 \quad \text{or}$$
$$-a(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$$

1) $abc + 2fgh - af^2 - bg^2 - ch^2$ is the **discriminant**.
2) If a=0 and b≠0, we do the above beginning with powers of y instead of x.
3) If a,b = 0, the method of factoring fails. But now we have

$$2hxy + 2qx + 2fy + c$$

and if this is resolvable to linear factors it must be of form

$$2h(x + p)(y + q)$$

∴ $2g = 2hq \quad 2f = 2hp \quad c = 2hpq$
The first two give $fg = h^2pq$ or $2hpq = 2fg/h$ ∴ $ch = 2fg$
$h \neq 0 \Rightarrow 2fgh - ch^2 = 0$ and when this is satisfied, resolution is

$$2hxy + 2gx + 2fy + c = 2h(x + f/h)(y + g/h)$$

4) If a,b,h = 0 we are left with $2gx + 2fy + c$
This is a linear factor and the discriminant vanishes.

Euclid's Algorithm

Let's review a little and then extend our idea of the GCM of polynomials. An ifn f(x) which exactly divides two or more given ifn of x is a **common measure** or factor or divisor of these ifns. The ifn of highest degree which exactly divides each of two or more such ifn is their **greatest common measure**.

If A = BQ + R: A,B,Q,R ifn of x, gcm(A,B) = gcm(B,R). And the proof of this takes exactly the form of the proof in DME where anything which divides the divisor and dividend, divides the divisor and remainder. I love that little proof.

In determining the GCM, we can add or remove to the remainder or divisor any ifn which has no common factor in them. We can remove a factor common to them both if we reintroduce it back into the GCM. We can add or remove a numerical factor to either. None of these affect the GCM.

With polynomials, we are using the degree of f(x) as a measure of magnitude. The larger polynomial has the larger degree. The proof of the ifn GCM is symmetric to the arithmetic proof except for its terminating in any constant and not just unity if the two fns are prime to each other (e.o.).

Thm. 2.7 $l,m,p,q \in \mathbf{N}$: $lq - mp \neq 0$ A,B,P,Q ifns:

$$P = lA + mB$$
$$Q = pA + qB$$

then gcm(P,Q) = gcm(A,B)

Proof

By inspection, any divisor of $A \wedge B$ divides $P \wedge Q$ ($\wedge \equiv$"and")

$qP - mQ = q(lA + mB) - m(pA + qB) = (lq - mp)A$

$-pP + lQ = -p(lA + mB) + l(pA + qB) = (lq - mp)B$

$\therefore \forall$divisor of $P \wedge Q$ divides $A \wedge B$ w/out remainder ■

This is sometimes called the Alternative GCD algorithm. As a purist, I consider it as a construction based upon Euclid 7.1. When using it, l,m,p,q are chosen so that the highest term in lA + mB and the lowest term in pA + qB disappear, just as one chooses numerical multipliers when solving simultaneous eqns using matrix arithmetic.

Examples

1) gcm(A,B)

$$A = 4x^4 + 26x^3 + 41x^2 - 2x - 24$$
$$B = 3x^4 + 20x^3 + 32x^2 - 8x - 32$$

$$-3A + 4B = 2x^3 + 5x^2 - 26x - 56$$
$$4A - 3B = 7x^4 + 44x^3 + 68x^2 + 16x$$

Don't be confused by this example. The l,m of -3,4 was used to make the x^4 cancel as $12x^4$'s. And it just happens that the same numbers can be used to cancel the -24 and -32. We can toss out the factor of x in the 2d one and continue:

$$A' = 7x^3 + 44x^2 + 68x + 16$$
$$B' = 2x^3 + 5x^2 - 26x - 56$$

Here comes another unfortunate coincidence for l,m,etc:

$$2A' - 7B' = 53x^2 + 318x + 424$$
$$7A' + 2B' = 53x^3 + 318x^2 + 424x$$

Again, toss the x factor and divide by 53:

$gcm(A,B) = x^2 + 6x + 8$

2) gcm(A,B)

$A = 2x^4 - 3x^3 - 3x^2 + 4$
$B = 2x^4 - x^3 - 9x^2 + 4x + 4$

Any divisor of A,B divides A-B or $-2x^3 + 6x^2 - 4x$
Or $-x(x^2 - 3x + 2)$ or $x(x - 1)(x - 2)$
By Remainder Thm, A,B divby both factors
$\therefore gcm(A,B) = x^2 - 3x + 2$

3) Sometimes there is no GCM
$A = x^2 - 3x + 1$
$B = x^2 - 4x + 6$
Using division with detached coeffs:

```
1 -3  1 | 1 -4  6
   2  1 |   -1  5
     11
```

This is Chrystal: *It will be well at this stage to caution the student against being misled by the analogy between the algebraical and the arithmetical GCM. He should notice that no mention is made of arithmetical magnitude in the definition of the algebraical GCM. The word "greatest" in that definition refers merely to degree. It is not even true that the arithmetical GCM of two arithmetical values of two given functions of x, obtained by giving x any particular value, is the arithmetical value of the GCM when that particular value of x is substituted therein; nor is it possible to frame any definition of the algebraical GCM so that this shall be true.*

Herr Niemand begs to differ. In example #1 above, the nvals are 70056 and 53088. And my calculator tells me that gcm(70056,53088) = 168 or, as x=10, $x^2 + 6x + 8$. So long as Niemand's Method remains a heuristic, I make no claims for it. But as it stands upon the form of number and the truth-grounds of arithmetic, no progress in its use will surprise me. In Chrystal's defense, being a Texan, I have an expensive Texas Instruments calculator which can factor 70056 in an instant and give me GCM's of any two integers that won't overrun the registers just as quickly. Done by hand, Niemand's Method would be mind-numbingly time-consuming.

Integral Functions as Integers

Because ifns are subject to Euclid's Algorithm, we have said they are "as integers." Let's see how far this analogy goes. The proofs of what follows are symmetric to the proofs in DME on number theory for integers.

Def. Two ifn are **prime to each other** (notation: p(A,B)) when they have no common divisor.

Thm. 2.8. ifns A,B ⇒ ∃ifns L,M prime to A,B:
gcm(A,B) = G ⇒ LA - MB = G and
p(A,B) ⇒ LA - MB = 1

You will see that last line again in partial fractions. The proof of this theorem is symmetric to the similar number theory proof. The same is true of the following theorem.

Thm. 2.9. ifns A,B,H p(A,B) ⇒ ∀common divisor of AH∧B divides H
Cor. 1. If AH divby B and p(A,B) ⇒ H divby B
Cor. 2. If A' prime to A,B,C,... ⇒ A' prime to their products
Cor. 3. If each of A,B,C,... prime to each of A',B',C',... then their products are prime to each other.
Cor. 4. m,n∈**N** If p(A,B) ⇒ $p(A^m,B^n)$
Cor. 5. If a set of ifns can be resolved into factors and powers of ifns A,B,C,... then the GCM of the set is the product of all factors common to all fns of the set, each factor raised to the lowest occurring power.

The take away here is that functions of form [ƒ] with coeffs ∈ **Z** are **as integers** and therefore, with due care, you can leverage the truths of number theory when dealing with them. What is true of one **must** be true of the other.

The Least Common Multiple also applies to ifns. Given a set of ifns, the LCM is the ifn of lowest degree divisible by each of them without remainder.

Let gcm(A,B) = G ⇒ A = aG ∧ B = bG where a,b ifns and p(a,b)
Let M be a common multiple of A,B ⇒ M = PA = PaG
M divby B = bG ⇒ M/bG = PaG/bG = Pa/b and is an ifn
∴ P = Qb where Q is an ifn ∴ M = QabG
∴ lcm(A,B) = abG = AB/G
∴ ∀common multiple of A,B is a multiple of lcm(A,B)

Integral Function Fractions

Def. If A,B ifns then A/B is an **integral function fraction**, denoted **ifrac**.

Note that in various textbooks, these can be called rational fractions or even rational functions. As always in mathematics, pay more attention to what it is than what it's called.

If A,B ifns, we say A < B if A° < B° and we can denote the degree of fractions where if A,B of degree m,n our notation is A_m/B_n. We can then distinguish between proper ifracs (pifrac) and improper ifracs (mifrac) with an exact analogy to numerical fractions. And again, as ifns are "as integers" many truths of ifracs follow from arithmetical or number theory proofs.

Thm. 3.1. ∀ifrac A/B: A > B is expressible as an ifn Q and an ifrac R/B
Proof follows from division algorithm using

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R}{B_n}$$

Cor. 1. If two mifracs A/B,A'/B' are equal then the integral and fractional parts are equal.
Proof

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R}{B_n} \quad \text{and} \quad \frac{A'_{m'}}{B'_{n'}} = Q_{m'-n'} + \frac{R}{B'_{n'}}$$

from this

$$Q_{m-n} - Q'_{m'-n'} = (R'B_n - RB'_{n'})/B_nB'_{n'} = C/D$$

degrees R,R' < n,n' ⇒ C° < n + n'
∴ if LHS≠0 ⇒ integral fn LHS = pifrac RHS ↴ (no integer can equal proper fraction)
∴ $Q_{m-n} = Q'_{m'-n'}$ ∴ $R/B_n = R'/B'_{n'}$ ■

Example using Niemand's Method

Show $\frac{x^3+2x^2+3x+4}{x^2+x+1} = \frac{x^5+4x^4+8x^3+12x^2+11x+4}{x^4+3x^3+4x^2+3x+1}$

LHS = $x + 1 + \frac{x+3}{x^2+x+1}$.

RHS = $x + 1 + \frac{x^3+5x^2+7x+3}{x^4+3x^3+4x^2+3x+1}$. nval = $\frac{1573}{13431} = \frac{11^2 \cdot 13}{3 \cdot 11^2 \cdot 37} = \frac{13}{111} \cdot \frac{11^2}{11^2}$

$= \frac{x+3}{x^2+x+1} \cdot \frac{(x+1)^2}{(x+1)^2}$ = LHS Note that using Niemand's Method always requires verification. Numerical factors can mislead.

Niemand works here because all coeffs are positive.

Thm. 3.2. ifns P,Q,P',Q' If P/Q = P'/Q' and P/Q in lowest terms then P' = λP Q' = λQ where λ ifn which reduces to a constant if P'/Q' is reduced to lowest terms.
Proof
P'/Q' = P/Q ∴ P' = Q'P/Q ⇒ LHS ifn ∴ Q'P divby Q
p(P,Q) ∴ Q' = λQ ∴ P' = λQP/Q = λP
∴ p(P',Q') ⇒ λ constant (could be unity)

Our ifracs can be operated on in the same fashion as numerical fractions. And, additionally, we can bring to bear all the tools of algebra. *Make every use you can of general ideas, such as homogeneity and symmetry to shorten work, to foretell results without labor, and to control results and avoid errors of the grosser kind.*

Examples

1) Using Niemand's Method

Solve $\dfrac{2x^3+4x^2+3x+4}{x^2+1} - \dfrac{2x^3+4x^2-3x-2}{x^2-1}$

Using nvals: $\dfrac{2434}{101} - \dfrac{2368}{99} = \dfrac{1798}{9999}$

Clearly 9999 is x^4-1. But the numerator must be determined. The negative coeffs in the RHT suggest that the first term in the num. is $2x^3$. By doing a quick and partial multiplication of detached coeffs for x^3, LHT is 3 + -2 = 1, RHT is -3 + 2 = -1 ∴ 1 - (-1) = 2 ∴ $2x^3$ verified. So we have $2x^3 + Ax^2 + Bx + C$. Using the same quick verification, we find A = -2 ∴ B=0 and C=2. So the num. is $2x^3 - 2x^2 - 2$ and 2000 - 202 = 1798.

2) Using Form

Simplify $F = \dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-c)(b-a)} + \dfrac{c^3}{(c-a)(c-b)}$

$= \dfrac{-a^3(b-c) - b^3(c-a) - c^3(a-b)}{(b-c)(c-a)(a-b)}$

Consider only the num. as fn. If b = c then fn = 0 ∴ divby (b - c)
By symm. divby (c - a) and (a - b)
Num. is 4° ∴ remaining factor is Pa + Pb + Pc ⇒ compare coeff of a^3b in num. and in P(a + b + c)(b - c)(c - a)(a - b) ∴ P = 1
∴ F = a + b + c

Partial Fractions

If we have ifracs A,B,C, we know we can add them "as integer" fractions where A+B+C = D and D will be an ifrac. So how do we go the other way, turning any D into a sum of A,B,C,...? We are going to drive this idea down as far as possible because it reveals a great deal about the form of number.

Thm. 3.3 pifrac A/PQ and p(P,Q) ⇒ A/PQ equals sum of pifracs P'/P + Q'/Q
Proof
p(P,Q) ⇒ ∃ifn L,M: LP + MQ = 1 (Thm. 1.4, 2.8)
∴ A/PQ = AL/Q + AM/P [1] (× A/PQ)
If RHS [1] mifracs, reduce:
AL/Q, AM/P = S + Q'/Q, T + P'/P: S,T ifns, Q'/Q,P'/Q pifracs
∴ A/PQ = S + T + Q'/Q + P'/P
LHS [1] pifrac ∴ S,T = 0
∴ A/PQ = Q'/Q + P'/P ■

Example

$$F = \frac{x^4 + 1}{(x^3 + 3x^2 + 2x + 1)(x^2 + x + 1)}.$$

∴ $A = x^4 + 1$ $P = x^3 + 3x^2 + 2x + 1$ $Q = x^2 + x + 1$

gcm(P,Q) = ?

```
            1 + 1 + 1 ) 1 + 3 + 2 + 1 ( 1 + 2 (= x + 2)
                            2   1   1
          (-x - 1 =)  -1   -1) 1  1  1 (-1 + 0
                               0   1
                                   1  ( = 1 )
```

∴ $R_1 = -x - 1$ $R_2 = 1$

GCM terminates in constant ∴ p(P,Q)

∴ $P = (x + 2)Q + R_1$ $Q = -xR_1 + R_2$

∴ $R_1 = P = (x+2)Q$

$R_2 = 1 = Q + xR_1$

$= Q + xP - x(x+2)Q$

∴ $1 = (-x^2-2x+1)Q + xP$

∴ $M = -x^2-2x+1$ $L = x$

∴ $A/PQ = ((x^4+1)(-x^2-2x+1))/P + ((x^4+1)x)/Q$

$$= \frac{-x^6 - 2x^5 + x^4 - x^3 - 2x + 1}{P} + \frac{x^5 + x}{Q}$$

$$= -x^3 + x^2 - 1 + \frac{x^2 + 2}{P} + x^3 - x^2 + 1 + \frac{-1}{Q} = \frac{x^2 + 2}{P} + \frac{-1}{Q}$$

Think of the theorem as justifying this brute-force method of attaining partial fractions. Then relax as we look at various cases and their abbreviated methods. We know that any ifn has real factors in the forms $(x - \alpha)^r$ and $(x^2 + \beta x + \gamma)^s$.

Case 1.

$F = A/B = A/((x-\alpha)Q)$: $(x-\alpha)$ is not a multiple root and p($(x-\alpha)$,Q)

∴ $F = P'/(x-\alpha) + Q'/Q$

$x - \alpha$ is 1° ∴ P' is 0° or constant

Let's do an example of this, noting that Q'/Q may itself be decomposable. Let us also note what an amazing example this actually is. Chrystal engineered it so that it simply continues into the other cases. I'm not the kind of person to go, "Oooh, that's a beautiful theorem" or any of that stuff. But I do admire simplicity and elegance and, especially, the expression of intelligence. And this is such an intelligently expressed example.

Example

$F = (4x^4-16x^3+ 17x^2 - 8x + 7)/((x-1)(x-2)^2(x^2+1)) = A/B$

∴ $F = p/(x-1) + Q'/((x-2)^2(x^2+1))$ and we need a constant p   [I]

∴ $A = p(x-2)^2(x^2+1) + Q'(x-1)$ [II]

Let x = 1 ∴ 4 = 2p ∴ p = 2

∴ $F = 2/(x-1) + Q'/((x-2)^2(x^2+1))$

Note that if, at this point, you need Q', in II let p = 2
and then subtract LHT of RHS from both sides
$\therefore 2x^4 - 8x^3 + 7x^2 - 1 = Q'(x-1)$
$\therefore$ LHS divby $(x-1) \Rightarrow Q' = 2x^3 - 6x^2 + x + 1$

Case 2.

$F = A/B = A/((x-\alpha)^r Q) : p((x-\alpha),Q)$
$\therefore F = P'/(x-\alpha)^r + Q'/Q \;\therefore P'$ ifn degree < r
$\therefore P' = a_0 + a_1(x-\alpha) + \cdots + a_{r-1}(x-\alpha)^{r-1}$
$\therefore F = a_0/(x-\alpha)^r + a_1/(x-\alpha)^{r-1} + \cdots + a_{r-1}/(x-\alpha) + Q'/Q$ where a_i are all constants

Example

Given same above $F = A/B = a_0/(x-2)^2 + a_1/(x-2) + Q'/(x-1)(x^2+1)$ [I]
$\therefore A = a_0(x-1)(x^2+1) + a_1(x-2)(x-1)(x^2+1) + Q'(x-2)^2$ [II]
In II let $x = 2 \therefore -5 = 5a_0 \therefore a_0=-1$
In II let $a_0=-1$, subtract LHT from both sides again, divide all by (x-2)
$\therefore 4x^3 - 7x^2 + 2x - 3 = a_1(x-1)(x^2+1) + Q'(x-2)$ [III]
In III let $x = 2 \therefore 5 = 5a_1 \therefore a_1=1$
$\therefore F = -1/(x-2)^2 + 1/(x-2) + Q'/((x-1)(x^2+1))$

Note that again Q' can be derived from this point using III.

Case 3.

$F = A/B = A/((x^2 + \beta x + \gamma)^s Q) : p((x^2 + \beta x + \gamma),Q)$
$\therefore F = P'/(x^2 + \beta x + \gamma) + Q'/Q : P'^\circ \leq 2s - 1$
So by expressing P' in powers of $(x^2 + \beta x + \gamma) = f$
$P' = (a_0 + b_0x) + (a_1 + b_1x)f + (a_2 + b_2x)f^2 + \cdots + (a_{s-1} + b_{s-1}x)f^{s-1}$
$\therefore F = (a_0 + b_0x)/f^s + (a_1 + b_1x)/f^{s-1} + \cdots + (a_{s-1} + b_{s-1}x)/(x^2 + \beta x + \gamma) + Q'/Q$

Note that if s = 1 then $F = (a_0 + b_0x)/(x^2 + \beta x + \gamma) + Q'/Q$

Example

Given same $F = A/B = (ax+b)/(x^2+1) + Q'/((x-1)(x-2)^2)$ [I]
$\therefore (4x^4 - 16x^3 + 17x^2 - 8x + 7)/(x^2+1) = ((ax+b)(x-1)(x-2)^2)/(x^2+1) + Q'$ [II]
By division, $4x^2 + 16x + 13 + (8x-6)/(x^2+1)$
$= (ax+b) \cdot (x + 5 + (7x+1)/(x^2+1)) + Q'$
$= (ax+b)(x-5) + (7ax^2 + (7b+a)x + b)/(x^2+1) + Q'$
$= (ax+b)(x-5) + 7a + ((7b+a)x + (b-7a))/(x^2+1) + Q'$ [III]
pifracs both sides of III are equal
$(7b+a)x + (b-7a) = 8x - 6$
$\therefore 7b+a = 8 \quad b-7a = 6 \therefore b = a = 1$
$\therefore F = (x+1)/(x^2+1) + Q'/Q$

OR

$x^2 + 1 = (x+i)(x-i) \therefore x^2 + 1 = 0$ when $x = i$

$\therefore 4x^4 - 16x^3 + 17x^2 - 8x + 7 = (ax+b)(x-1)(x-2)^2 + Q'(x^2+1)$

$= (ax+b)(x^3 - 5x^2+8x - 4) + Q'(x^2+1)$

Let $x = i \therefore 8i - 6 = (ai+b)(7i + 1)$

$= (7b+a)i + (b - 7a)$

$\therefore (7b + a - 8)i = -b + 7a - 6$

which is impossible unless both sides equal zero

$\therefore 7b + a - 8 = 0$ and $7a - b - 6 = 0$ and algebrate

OR

Suppose we did the first cases and have:

$F = 2/(x-1) - 1/(x-2)^2 + 1/(x-2) + (ax+b)/(x^2+1)$

$\therefore 4x^4 - 16x^3 + 17x^2 - 8x + 7 =$

$2(x-2)^2(x^2+1) -(x-1)(x^2+1) + (x-1)(x-2)^2(x^2+1) + (ax+b)/(x-1)(x-2)^2$

$\therefore x^4 - 4x^3 + 3x^2 + 4x - 4 = (ax+b)(x-1)(x-2)^2$

$\therefore$ LHS divby RHS factors $\therefore x + 1 = ax + b \therefore a = b = 1$

What an example. Every example should be that good.

I first met partial fractions in a calculus review book. It was written by an engineer and he was soooo proud of his partial fractions. He did them with the derivatives and he did them with the integrals and given half an excuse he would have done them again. But you could tell that he didn't really know what he was doing. It was rougher than Niemand's Method: half heuristic, half guessing game. It ticked me off. So I studied partial fractions wherever I could find them. And like continued fractions or taijiquan, the road goes on and on as far as you want to take it. Some roads never end. Let's see how far we can go without making this a book on partial fractions. We'll just check out some of the good stuff.

Example

1)

Another general method for this is equating coeffs.

Decompose $(3x-4)/((x-1)(x-2))$

$\therefore = a/(x-1) + b/(x-2) \therefore 3x - 4 = a(x-2) + b(x-1)$

$\therefore 3x - 4 = (a+b)x - (2a+b)$

$\therefore a + b = 3 \quad 2a + b = 4 \therefore a=1\ b=2$

2)

Or we could use the form of number

$F = (x^2 + px + q)/((x-a)(x-b)(x-c)) = A/(x-a) + B/(x-b) + C/(x-c)$ [I]

$\therefore x^2 + px + q = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$ [II]

Let $x=a \therefore a^2 + pa + q = A(a-b)(a-c)$

$\therefore A = (a^2 + pa + q)/(a-b)(a-c)$ Sym. for B,C

$\therefore F = (a^2 + pa + q)/((a-b)(a-c)(x-a)) + (b^2 + pb + q)/((b-c)(b-a)(x-b))$

$+ (c^2 + pc + q)/((c-b)(c-a)(x-c))$

The method of proper fractions is in point of fact a fruitful source of complicated algebraic identities.

We can tie this idea of partial fractions to series.
Required: the series expansion of $(2x - 3)/(x^2 - 3x + 2)$
By partial fractions, $(2x - 3)/(x^2 - 3x + 2) = 1/(x-1) + 1/(x-2) = -1/(1-x) - 1/(2-x)$
$-1/(1-x) = -(1-x)^{-1} = -(1 + x + x^2 + \cdots + x^n + \cdots)$
$-1/(2-x) = -½(1 - x/2)^{-1} = -½(1 + x/2 + x^2/2^2 + \cdots + x^n/2^n + \cdots)$
∴ general term of $(2x - 3)/(x^2 - 3x + 2)$ is $-(1 + 1/2^n)x^n$

Let us show that the coeffs of such a series are connected by a determinable relation. Recall our theorem from DMS that if two series are equal for any finite x then the coeff of corresponding terms must be equal. If these are equal

$$\alpha = a_0 + a_1x + a_2x^2 + \cdots$$
$$\beta = A_0 + A_1x + A_2x^2 + \cdots$$

Then $a_0 - A_0 = 0$, $a_1 - A_1 = 0$, ... So the sum $\alpha+\beta = 0$. Importantly, this is true only of x: $\alpha+\beta$ convergent because a divergent series need not vanish.

Let $(a + bx)/(1 - px - qx^2) = u_0 + u_1x + u_2x^2 + \cdots$
∴ $a + bx = (1 - px - qx^2)(u_0 + u_1x + u_2x^2 + \cdots)$
If $n > 1$, coeff of x^n on RHS is $u_n - pu_{n-1} - qu_{n-2}$ (by DME infinite series multiplication)
But x on LHS is first power ∴ $n > 1$, $u_n - pu_{n-1} - qu_{n-2} = 0$ [1]
Using 1st and 2d terms both sides: $u_0 = a$ $u_1 - pu_0 = b$
Then we can use [1] and these to determine $u_2, u_3, \ldots$ by making $n = 2,3,\ldots$

Recall our fn notation from DME and we have a shorter general proof of partial fractions from some guy named Cox via Todhunter's *Integral Calculus*. I love short and simple.

Let $F(x) = (x-a)^n\psi x$

$$\therefore \frac{\varphi x}{Fx} = \frac{\varphi x}{(x-a)^n\psi x} = \frac{\varphi x - (\varphi a/\psi a)\psi x}{(x-a)^n\psi x} + \frac{(\varphi a/\psi a)}{(x-a)^n}$$

Numerator LHT RHS = 0 when x=a ∴ divby (x-a)
Let this quotient = χx

$$\therefore \frac{\varphi x}{Fx} = \frac{\chi x}{(x-a)^{n-1}\psi x} + \frac{\varphi a}{\psi a} \cdot \frac{1}{(x-a)^n}$$

This process can be repeated on LHT RHS until it terminates ■
And you will see that again and again before we are done.
Note in this that if $a = \alpha+\beta i$ then $b = \alpha - \beta i$ is also a root. So if we add the pifracs $\varphi a/\psi a \cdot 1/(x-a)^n + \varphi b/\psi b \cdot 1/(x-b)^n$ then the result is free of i.
Since we know derivatives from DME, consider

Todhunter's Case 1:

$\varphi x/Fx = A/(x-a) + \chi x/\psi x$ [1]
∴ $\varphi x = A\psi x + (x-a)\chi x$ [2]
Let x = a in [2] ∴ $\varphi a = A\psi a$ ∴ $A = \varphi a/\psi a$
$Fx = (x-a)\psi x$ ∴ $F'x = \psi x + (x-a)\psi' x$ ∴ $F'a = \psi a$
∴ $A = \varphi a/F'a$

Todhunter's Case 2:

Let $Fx = (x-a)^n \psi x$

$\therefore\ \varphi x/Fx = A_1/(x-a)^n + A_2/(x-a)^{n-1} + \cdots + A_n/(x-a) + \chi x/\psi x$

Multiply both sides by $(x-a)^n$ then let $f(x) = \varphi x/Fx \cdot (x-a)^n$

$f(x) = A_1 + A_2(x-a) + A_3(x-a)^2 + A_n(x-a)^{n-1} + \chi x/\psi x \cdot (x-a)^n$

Differentiate both sides **and then** let x=a and we have (keep in mind what f(x) is):

$f(a) = A_1$

$f'(a) = A_2$

$f''(a) = 2!A_3$

$f'''(a) = 3!A_4$

…

$f^{n-1}(a) = (n-1)!A_n$

and we have determined A_i.

Where Chrystal restricts his factors to those with real coeffs, Todhunter's cases 3 and 4 are i-roots, single and multiple.

Todhunter's Case 3:

Partial fractions for a single pair of i-roots, $\alpha \pm \beta i$

Decompose $\varphi x/Fx$. Roots $a + bi \Rightarrow$ partial fractions are

$\varphi a/F'a \cdot 1/(x-a)$ and $\varphi b/F'b \cdot 1/(x-b)$

Let these be $(A - iB)/(x-a)$ and $(A + iB)/(x-b)$ or $(A-ib)/(x-\alpha-\beta i)$ and $(A+iB)/(x-\alpha+\beta i)$

And their sum is $(2A(x-\alpha) + 2B\beta)/((x-\alpha)^2 + \beta^2)$

OR

Let $x^2 - px + q$ have our roots $\alpha+\beta i$ then

$$\varphi x/Fx = (Lx + M)/(x^2 - px + q) + \chi x/\psi x$$

$\therefore Fx = (x^2 - px + q)\psi x$ (× Fx)

$\therefore \varphi x = (Lx + M)\psi x + (x^2 - px + q)\chi x$ [1]

Let x = either root ∴ [1] becomes

$$\varphi x = (Lx + M)\psi x \qquad [2]$$

Repeatedly sub px-q for x^2 in [2] until eqn 1° in x with form $Px + Q = P'x + Q'$

Let $x = \alpha+\beta i$ and equate i-coeffs then we have 2 simult. eqns as in 2d method of Case 3 above.

Todhunter's Case 4:

Let $x^2 - px + q$ have roots again of $\alpha+\beta i$ and let this quadratic factor occur r times. Solvable by method of Case 2.

OR

$$\frac{\varphi x}{Fx} = \frac{L_r x + M_r}{(x^2-px+q)^r} + \frac{L_{r-1}x + M_{r-1}}{(x^2-px+q)^{r-1}} + \cdots + \frac{L_1 x + M_1}{x^2-px+q} + \frac{\chi x}{\psi x}$$

Multiply both sides by $Fx = (x^2-px+q)^r\psi x$

$\therefore \varphi x = (L_r x + M_r)\psi x + (L_{r-1}x + M_{r-1})(x^2-px+q)\psi x + \cdots + (x^2-px+q)^r\chi x$ [1]

Let x = either root $\therefore \varphi x = (L_r x + M_r)\psi x$
Find L_r, M_r as in Case 3. Then from [1]

$$\varphi x - (L_r x + M_r)\psi x = (L_{r-1}x + M_{r-1})(x^2-px+q)\psi x + \cdots$$

$\therefore$ quad factor both sides. Divide by quad factor. Let LHS = $\varphi_1 x$
$\therefore \varphi_1 x = (L_{r-1}x + M_{r-1})(x^2-px+q)\psi x + \cdots$ and proceed as above to find L_{r-1}, M_{r-1} until all L_i, M_i determined.

Todhunter's cases 3,4 are not that different than Chrystal's. It is a matter of reading the mathematics at a somewhat higher level. And then the question becomes, Can you express the mathematics practically in a partial fraction from the higher standpoint?

Just one more idea with partial fractions and then we'll move on. You have basic integral Calculus from DME. So you can now take the integral of anything like

$$(5x^2+1)/x^2 - 3x + 2)$$

By division = $5x + 15 + (35x - 29)/(x^2 - 3x + 2)$
RHT,RHS = $A/(x-1) + B/(x-2) \therefore 35x - 29 = A(x-2) + B(x-1)$
Let x = 1 and then 2
$\therefore 35 - 29 = -A \therefore A = -6 \quad 70 - 29 = B \therefore B = 41$
$\therefore F = 5x + 15 - 6/(x-1) + 41/(x-2)$
$\therefore \int F\,dx = 5x^2/2 + 15x - 6\ln(x-1) + 41\ln(x-2)$

3. Algebraic Functions

A few remarks on the way. Consider fractional powers $k^{1/p}$:
If $k = h^{2p}$ then $k^{1/2p}$ has two real values: $\pm h$.
If $k = h^{2p+1}$ then $k^{1/(2p+1)}$ has one real value: h.
If $k = -h^{2p+1}$ then $k^{1/(2p+1)}$ has one real value: -h.

Consider also that with $x^{p/q}$, p/q must be an actual fraction in lowest terms, else

$$x^{4/2} = \sqrt{x^4} = \pm x^2$$
$$\therefore x^2 = -x^2$$

OR

$$(x^m)^n = x^{mn} = (x^n)^m$$
$$(x^{½})^2 = (x^2)^{½}$$
$$\therefore (\pm\sqrt{x})^2 = \pm x$$
$$x = \pm x$$

Putting p/q in lowest terms avoids this error, which is an obvious error here but not so obvious when the expressions are more complex.

Rationalizing Factors

Def. P,Q ifns of given irrationals (constants or variables with fractional coeffs). If PQ is rational wrt the given irrationals then Q is a **rationalizing factor** of P wrt the given irrational monomials.

Example

$$P = A(p^{½})^{2m+1}(q^{½})^{2n+1} = (Ap^m q^n)p^{½}q^{½}$$
$$\text{Let } Q = p^{½}q^{½} \; \therefore PQ = (Ap^m q^n)pq$$

$$P = 16 \cdot 2^{3/2} \cdot 3^{5/2} \cdot 5^{1/2}$$
$$Q = 2^{1/2} \cdot 3^{1/2} \cdot 5^{1/2} = 30^{1/2} \; \therefore \; PQ = 16 \cdot 2^2 \cdot 3^3 \cdot 5$$

$$P = Ap^{l/s}q^{m/t}r^{n/u}$$
$$Q = p^{1-l/s}q^{1-m/t}r^{1-n/u} \; \therefore \; PQ = Apqr$$

In DME we talked briefly about tossing out the radicals to simplify an equation. And that is what we are doing here but more methodically. To rationalize binomials:

1) $\forall a\sqrt{p} + b\sqrt{q}$ rationalizing factor (ratfac) is $a\sqrt{p} - b\sqrt{q}$
which is a use of $(a+b)(a-b) = a^2-b^2$ or here $a^2p - b^2q$.
These factors are **conjugates**.

2) $ap^{\alpha/\gamma} \pm bq^{\beta/\delta}$ Let $x = ap^{\alpha/\gamma}$ $y = bq^{\beta/\delta}$
$P = ap^{\alpha/\gamma} - bq^{\beta/\delta} = x - y$
$m = \text{lcm}(\gamma,\delta)$
$\therefore (x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1})P = x^m - y^m$
$= a^m p^{m\alpha/\gamma} - b^m q^{m\beta/\delta}$ where $m\alpha/\gamma, m\beta/\delta \in \mathbf{Z} \therefore x^m - y^m$ rational
$\therefore (x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}) \equiv$ ratfac

Trinomials of form $P = \sqrt{p} + \sqrt{q} + \sqrt{r}$ are rationalized using conjugates.

$$(\sqrt{p} + \sqrt{q} - \sqrt{r})P = (\sqrt{p} + \sqrt{q})^2 - (\sqrt{r})^2 = p + q - r + 2\sqrt{pq}$$

Using conjugates again:

$$(p + q - r - 2\sqrt{pq})(\sqrt{p} + \sqrt{q} - \sqrt{r})P = (p + q - r)^2 - (2\sqrt{pq})^2$$
$$= p^2 + q^2 + r^2 - 2pq - 2pr - 2qr$$

Note that the ratfac here is:

$$(\sqrt{p} + \sqrt{q} + \sqrt{r})(\sqrt{p} + \sqrt{q} - \sqrt{r})(\sqrt{p} - \sqrt{q} - \sqrt{r})$$

which is every permutation of signs in $\sqrt{p} \pm \sqrt{q} \pm \sqrt{r}$ which is not in P. This use of perms is the general rule. Given $1 + \sqrt{2} + \sqrt{3} + \sqrt{5}$ the signs are ++++. So we would need $2^3 - 1$ factors where the first term is always +.

Thm. 3.1. ∀ifn of square roots $\sqrt{p},\sqrt{q},\ldots$ can be expressed as a sum of rational terms and rational multiples of $\sqrt{p},\sqrt{q},\ldots$ and their products $\sqrt{pq},\ldots$
Proof
∀ifn $\varphi(\sqrt{p})$: $\sqrt{p}$ only square root
∀term of even° rational, ∀term $(2m+1)° = A(\sqrt{p})^{2m+1}$
which reduces to $(Ap^m)\sqrt{p}$. Rational to P, roots to Q
$\therefore \varphi(\sqrt{p}) = P + Q\sqrt{p}$ [1]
∀ifn $\varphi(\sqrt{p},\sqrt{q})$ then by [1] $\varphi(\sqrt{p},\sqrt{q}) = P + Q\sqrt{p}$
Sym. $P = P' + Q'\sqrt{q}$ and $Q = P'' + Q''\sqrt{q}$
$\therefore \varphi(\sqrt{p},\sqrt{q}) = P' + Q'\sqrt{q} + (P'' + Q''\sqrt{q})\sqrt{p}$
$= P' + P''\sqrt{p} + Q'\sqrt{q} + Q''\sqrt{pq}$
and so on for $\varphi(\sqrt{p},\sqrt{q},\sqrt{r})\ldots$ ■
Cor. 1.
$\varphi(-\sqrt{p}) = P - Q\sqrt{p}$ $\therefore \varphi(-\sqrt{p})$ is ratfac of $\varphi(\sqrt{p})$
Cor. 2.
$\forall\varphi(\sqrt{p},\sqrt{q},\sqrt{r},\ldots)$ then for any root, say $\sqrt{q}$, $\varphi(\sqrt{p},-\sqrt{q},\sqrt{r},\ldots)$ is a ratfac.

Example

$\varphi x = x^3 + x^2 + x + 1$
$\varphi(1+\sqrt{3}) = 16 + 9\sqrt{3}$
$\therefore \varphi(1-\sqrt{3})$ is the conjugate of $\varphi(1+\sqrt{3})$ or $16 - 9\sqrt{3}$
$\therefore \varphi(1+\sqrt{3})\varphi(1-\sqrt{3}) = 16^2 - (9\sqrt{3})^2 = 13$

Theorem 3.1 can be used to prove the general rule for ratfacs. For every perm of signs, we must have its conjugate. You need one conjugate for the first root, two for the second, four for the third, ... or given n terms, 2^n-1 conjugates.

Thm. 3.2. ∀fractional fn, ifn or not, of √p,√q,... can be expressed as a sum of the rational part and rational multiples of roots and their products.
Proof
R,P of √p,√q,..., Q ratfac of P
∴ R/P = RQ/PQ where PQ rational and RQ ifn of √p,√q,...
∴ RQ expressible in required form by Thm. 3.1. ■

Example

$$\frac{1}{1+\sqrt{2}+\sqrt{3}} = \frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2} = \frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}(1+\sqrt{2}-\sqrt{3}}{4}$$

$$= \tfrac{1}{4}(\sqrt{2}+2-\sqrt{6}) = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2} - \tfrac{1}{4}\sqrt{6}$$

It can be shown that what is true above for square roots is true of any fractional exponents and that a ratfac always exists.

Surds

Nowadays surds are called **radicals**. But like "greatest common measure", I stick with the old word which is the Latin word for "root." A surd is the root of any fn with coeffs in **Q** when these roots contain values from **Q** with fractional exponents which are not imaginary. So 2+√ε is not a surd but 1+√2 is a surd. Surds are **algebraic numbers** and are constructible in Euclidean geometry. If you can construct √2 + 1, you can construct √(1 + √(2 + 1)) and so on to any depth. Although in older texts, these compound surds are not considered surds, they just are. Simply stated, surds are radicals, are algebraic numbers, are the roots of functions with coeffs in **Q**.

Def. Surds are **similar** (~) when they can be expressed as multiples of the same surd.

Example

$\sqrt{18} = 3\sqrt{2}$ $\sqrt{8} = 2\sqrt{2}$ $\therefore$ $\sqrt{18} \sim \sqrt{8}$ $\sqrt{6}$!~ $\sqrt{8},\sqrt{18}$

Def. Surds arising from the same nth root are **equiradical** (eqR). There are n-1 surds eqR with $p^{1/n}$: $p^{1/n}$, $p^{2/n}$, ..., $p^{(n-1)/n}$. Let m > n, then $p^{m/n} = p^{\lambda+\mu/n}$ = rational multiple of one of $p^{1/n}$'s equiradicals. All surds eqR with $p^{1/n}$ are ifn of $p^{1/n}$ and ∀fn of $p^{1/n}$ is expressible as a linear function of these n-1 eqR surds in the form:

$$A_0 + A_1 p^{1/n} + A_2 p^{2/n} + \cdots + A_{n-1} p^{(n-1)/n}$$

where $\forall A_i$ are rational wrt $p^{i/n}$. The product and quotient of two similar quadratic surds are rational and conversely. If surds are roots of a function of degree > 2, the product of two similar or eqR surds is either rational or an eqR surd.

Thm. 3.3. If p,q,A,B ∈ **Q**, A ≠ 0, √p,√q ∈ **R-Q** (those reals which are not rational) we cannot have √p = A + B√q.

Proof

Else $p = (A + B\sqrt{q})^2 = A^2 + B^2 q + 2AB\sqrt{q}$

∴ $q = (p - A^2 - B^2 q)/2AB$ ↴ (LHS∈**R-Q** RHS∈**Q**) ■

In ratfacs, we saw that any fn of √q has the form A + B√q, ∴ one quadratic surd cannot be expressed as a rational fn of a dissimilar surd or

$$\text{If } \sqrt{p} = A + B\sqrt{q} \text{ then } \exists m \in \mathbf{Q}: \sqrt{p} = m\sqrt{q}$$

It follows that **no rational solution** which is not a mere eqn ($(\sqrt{3})^2+(\sqrt{2})^2 = 5$) can exist between two dissimilar surds. This means the relation must be in **C-Q**.

∴ A quadratic surd cannot be a rational fn of two dissimilar quadratic surds.

∴ A quadratic surd cannot be the sum of two dissimilar quadratic surds.

It follows that if A + B√p + C√q + D√pq = 0 then A,B,C,D = 0

Thm. 3.4. x,y,z,u ∈ **Q** √y,√u ∈ **R-Q** x+√y = z+√u ⇒ x=z, y=u

Proof

x = a+z: a≠0

∴ a+z+√y = z+√u

∴ a+√y = √u

∴ $a^2 + y + 2a\sqrt{y} = u$

∴ $\sqrt{y} = (u - a^2 - y)/2a$ ↴ (√y !∈ **Q** by hyp.)

∴ x=z ∴ y=u ■

Let's look deeper into taking square roots of simple surds to see whether the results can be expressed as surds:

$$\sqrt{(p+\sqrt{q})} \qquad \sqrt{(p - \sqrt{q})}$$

From DME, we know we can set √(p+√q) = √x + √y or √(p - √q) = √x - √y. We take the first and square both sides:

$$p + \sqrt{q} = x + y + 2\sqrt{xy}$$
$$\therefore x+y = p \qquad [1]$$
$$\text{and } 2\sqrt{xy} = q \qquad [2]$$

Squaring p and subtracting 4xy we have

$$(x+y)^2 - 4xy = p^2 - q$$
$$\therefore (x - y)^2 = p^2 - q$$
$$\therefore x - y = \pm\sqrt{(p^2 - q)} \qquad [3]$$

With upper sign, using [3],[1]

$$(x + y) + (x - y) = p + \sqrt{(p^2 - q)}$$
$$(x + y) - (x - y) = p - \sqrt{(p^2 - q)}$$

Adding and subtracting

$$2x = p + \sqrt{(p^2 - q)} \qquad 2y = p - \sqrt{(p^2 - q)}$$
$$\therefore x = (p + \sqrt{(p^2 - q)})/2 \qquad y = (p - \sqrt{(p^2 - q)})/2$$

Note that using the lower sign in [3] merely exchanges x and y

$$\therefore \sqrt{x} = \pm\sqrt{(\ (p + \sqrt{(p^2 - q)})/2\ )} \qquad \sqrt{y} = \pm\sqrt{(\ (p - \sqrt{(p^2 - q)})/2\ )}$$

Note that in [2] $2\sqrt{xy} = 2\sqrt{x}\cdot\sqrt{y} = +q \therefore \sqrt{x} = \pm\ldots\ \sqrt{y} = \pm\ldots$ above. If we had begun with $\sqrt{(p - \sqrt{q})}$ these would be opposites ($\pm\ldots, \mp\ldots$) to give -q.

$$\therefore \sqrt{(p + \sqrt{q})} = (\ \sqrt{((p + \sqrt{(p^2 - q)})/2)} + \sqrt{((p - \sqrt{(p^2 - q)})/2)}\ )$$
$$\sqrt{(p - \sqrt{q})} = (\ \sqrt{((p + \sqrt{(p^2 - q)})/2)} - \sqrt{((p - \sqrt{(p^2 - q)})/2)}\ )$$

Compare this to our $\sqrt{(11 + 4\sqrt{7})} = 2 + \sqrt{7}$ in DME. We can only get a simple surd here when $p^2 - q$ is a perfect$^2 \in \mathbf{N}$. Here $11^2 - 16\cdot 7 = 9$. We are studying the form of surds, looking down on it.

When can $\sqrt{(p + \sqrt{q} + \sqrt{r} + \sqrt{s})}$ take the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$?
Set them equal and square:

$$p + \sqrt{q} + \sqrt{r} + \sqrt{s} = x + y + z + 2\sqrt{yz} + 2\sqrt{zx} + 2\sqrt{xy}$$

Let $2\sqrt{yz}, 2\sqrt{zx}, 2\sqrt{xy} = \sqrt{q}, \sqrt{r}, \sqrt{s}$ Then multiplying the last two:

$$4x\sqrt{yz} = \sqrt{rs}$$

Combined with the first:

$$x = \tfrac{1}{2}\sqrt{(rs/q)}$$

Sym. $y = \tfrac{1}{2}\sqrt{(qs/r)}$, $z = \tfrac{1}{2}\sqrt{(qr/s)}$
And from the original forms $x + y + z = p$
$\therefore \sum(\sqrt{rs/q}) = 2p$ where each term must be positive
$\therefore$ each square root must be rational
$\therefore rs/q = \alpha^2 \quad qs/r = \beta^2 \quad qr/s = \gamma^2$ where $\alpha,\beta,\gamma \in \mathbf{Q}$ and $\alpha+\beta+\gamma = 2p$
$\therefore$ In order to take form $\sqrt{x}+\sqrt{y}+\sqrt{z}$: $q = \beta\gamma$, $r = \gamma\alpha$, $s = \alpha\beta$

Recall from DME that when we take the square root of a number that you can double the decimal places at any point by dropping the square root algorithm and continuing by using division although the last digit may be wrong. Let's look at why this works.

Thm. 3.5. N∈**N**, if the first p of n digits of $\sqrt{N}$ are found $d_1d_2...d_p = P$: $P\cdot 10^{n-p} \cong \sqrt{N}$ then the next p-1 digits will be the first p-1 digits of the integral part of the quotient:

$$\frac{N - (P\cdot 10^{n-p})^2}{2\cdot P\cdot 10^{n-p}}$$

Proof

Let the rest of the square root's digits make up the number Q

$\therefore \sqrt{N} = P\cdot 10^{n-p} + Q$ where $10^{p-1} < P < 10^p$ and $Q < 10^{n-p}$

$\therefore N = (P\cdot 10^{n-p})^2 + 2PQ10^{n-p} + Q^2$

$\therefore \frac{N - (P10^{n-p})^2}{2P10^{n-p}} = Q + \frac{Q^2}{2P10^{n-p}}$

where $\frac{Q^2}{2P10^{n-p}} < \frac{10^{2(n-p)}}{2} \cdot 10^{p-1} \cdot 10^{n-p} < \frac{10^{n-2p+1}}{2}$

$\therefore Q^2/2P10^{n-p}$ affects at most the (n-2p)th decimal place from the right and Q has n-p digits.

$\therefore (n-p) - (n - 2p + 1) = p - 1$ digits in quotient from left hand side are correct. ■

You can take a calculator, a big n∈**N**, find $\sqrt{n}$ and work out an example of this for yourself. Think about how this proof uses disparate forms of number, including the form of positional decimal notation, to achieve its end. Let's do that again with ifns.

Thm. 3.6. If $F = p_0x^{2n} + p_1x^{2n-1} + \cdots$ and $\sqrt{F} = (q_0x^n + q_1x^{n-1} + \cdots + q_{n-\gamma+1}x^{n-\gamma+1}) + (q_{n-p}x^{n-p} + \cdots + q_n) = P + Q$ and if the first p terms of $\sqrt{F} = P$ are known then the next p terms are the first p terms of the integral part of $(F - P^2)/2P$.

Proof

$F = P^2 + 2PQ + Q^2$

$\therefore (F - P^2)/2P = Q + Q^2/2P$

$\therefore Q^2/2P$ is degree $2(n-p)-n = n - 2p$

$\therefore$ down to term x^{n-2p+1} $(F - P^2)/2P = Q$

or the first $n - p - (n - 2p) = p$ terms LHS are first p terms of Q for $\sqrt{F}$ ■

Example

We do this just like the integer algorithm in DME for the first three terms then we use the above theorem for the next three in taking $\sqrt{F}$. Note that the doubling of the "divisor" only affects the latest of its terms.

$F = x^{10} + 6x^9 + 13x^8 + 4x^7 - 18x^6 - 12x^5 + 14x^4 - 12x^3 + 9x^2 - 2x + 1$

```
               1  6  13  4  -18  -12  14  -12  9  -2  1 | 1
            1 ) 1
        2 + 3 ) 6  13  4  -18  -12  14  -12  9  -2  1 | 3
                6   9
    2 + 6 + 2 ) 4   4  -18  -12  14  -12  9  -2  1 | 2
                    4  12    4
2 + 6 + 4 + -4 ) -8  -22  -12  14  -12  9  -2  1 | -4        ( -8÷2 )
                 -8  -24  -16  16
       2 + 6 + 4 + 8 ) 2    4   -2  -12  9  -2  1 | 1         ( 2÷2 )
                       2    6    4   -8
                           -2   -6   -4  9  -2  1 | -1        ( -2÷2 )
                           -2   -6   -4  8
                                         1  -2  1
```

∴ $\sqrt{F} = x^5 + 3x^4 + 2x^3 - 4x^2 + x - 1$ and because the remainder $x^2 - 2x + 1$ is the square of the last two terms $x - 1$, F is a perfect². And by changing all the signs in $\sqrt{F}$ we get the other square root. The nval of F here is 17320928881 so the √nval = 131609. But the negative coeffs confuse the issue after $x^5 + 3x^4$. We could start with the √nval and work this out.

If we have $\sqrt{F} = x^5 + px^4 + qx^3 + rx^2 + sx + t$ then
$F = x^{10} + 2px^9 + (p^2+2q)x^8 + (2pq + 2r)x^7 + \cdots$ where
$2p = 6 \quad p^2 + 2q = 13 \quad 2pq+2r = 4$ all from F
∴ $p=3 \quad q=2 \quad r=-4$
∴$\sqrt{F} = x^5 + 3x^4 + 2x^3 - 4x^2$ with nval 131600
∴ $0x + 9$ or $1,-1$ ∴ $x - 1$ for last two terms.

This combines using the nval with our nontrivial indeterminate coeff theorem from DME. Of course, if F is not a perfect² of an ifn, $\sqrt{F}$ becomes an approximation. But this is revealed immediately when you take the square root of the nval of F.

Just as we can take a non-perfect² and use the square root algorithm to approximate a result and just as we can divide 1 by 1-x to get $1 + x + x^2 + x^3 + \cdots$ by long division, we can use this algorithm for $\sqrt{F}$ to produce an infinite series for $\sqrt{(x+1)}$ and similar. But a grain of compassion remains in my mathematically-hardened heart so we won't do that here. But it will hurt so good if you do it yourself.

Complex Numbers

Thm. 3.7. The sum, difference, product, and quotient of $\forall z_1, z_2 \in \mathbf{C}$ are also complex numbers in **C** with form a+bi: $a,b \in \mathbf{R}$.

We demonstrated this theorem in DME. In this text, we will only include the more general ideas of complex numbers and leave the more developed ideas for the Second Circle of Trigonometry.

A couple of definition reminders:

A **real fn** is a fn of form [*f*] with coeffs ∈ **R**.

A **rational fn** is a fn of form [*f*] with coeffs ∈ **Q**.

Cor. 1. Because every rational fn (ifn) involves only the four operation of arithmetic, every such fn of one or more complex vars produces complex numbers.
Cor. 2. ∀real fn f(x+iy) can be reduced to fns P + Qi where P,Q real fns. P contains only even powers of y, Q only odd powers. If we change the sign of y, Q changes sign and P does not ∴ f(x+iy) = P + Qi ⇔ f(x-iy) = P - Qi.
Cor. 3. If $\varphi(x_1+iy_1,\ldots,x_n+iy_n)$ is a real fn of n complex numbers = X + iY then $\varphi(x_1- iy_1,\ldots,x_n- y_n)$ = X - iY where X,Y real fns.
Cor. 4. If φ(z) vanishes for z = a + bi then φ vanishes for z = a - bi.
Cor. 5. If ∀real fn $\varphi(x_1+iy_1,\ldots,x_n+iy_n)$ vanishes for n values of $x_i + iy_i$ then φ vanishes if all those values are replaced by their conjugates.

Cor. 2. is extremely useful. Any real fn of complex vars **always** reduces to a real term and an imaginary term. This is a basic truth like "any fn of degree n has n roots".

Example with our ∑ notation combs

$$\begin{aligned}&(b + c - ai)(c + a - bi)(a + b - ci)\\ &\quad= (\prod(b + c) - \sum c(b + c)) + (abc - \sum a(a + b)(a +c))i\\ &\quad= 2abc + (abc - \sum a^3 - \sum a^2(b + c) - 3abc)i\\ &\quad= 2abc - (\sum a^3 + (b + c)(c + a)(a + b))i\end{aligned}$$

Example of Binomial Term of Any Exponent

$$(x + yi)^4 = (x^4 - 6x^2 + y^4) + (4x^3y - 4xy^3)i$$

Even powers of y turn i into alternating ±1, odd into alternation ±i.

We know that a+bi and a-bi are conjugates. Their sum (2a) and product (a^2+ b^2) are real. Conversely, if the sum and product of two complex numbers are real then they are either conjugates or they are purely real.

Def. The **norm** of x + yi is x^2+y^2, denoted "norm".
Def. The **modulus** of x + yi is $\sqrt{(x^2+y^2)}$, denoted "mod".

Warning: These terms vary in use. Some texts equate them. In some algebras, the norm is generalized into "inner product" and then the modulus becomes the norm. Here's the important part: the one under the radical sign gives the magnitude of x+iy. Mathematicians will never get their name-space sorted. So pay attention to what each thing actually **is** and let each book call it whatever it wants to. We'll stick with the above def.s in this text.

Conjugates have the same norm and the same modulus. If a complex number vanishes, its modulus vanishes and conversely. If two complex numbers are equal, their moduli are equal, but **not** conversely. Or $x^2+y^2 = x'^2+y'^2 \;!\Rightarrow x=x' \wedge y=y'$.

Thm. 3.8. ∀real φ, norm(φ(a+bi)) = norm(φ(a-bi)) = φ(a+bi)φ(a-bi)

Proof

φ(x+yi) = X + Yi ∴ φ(x-yi) = X - Yi

∴ norm(φ(x+yi)) = norm(X - Yi) = X^2+Y^2

= (X + Yi)(X - Yi) = φ(x+yi)φ(x-yi) ■

Cor. 1. mod(φ(a+bi)) = mod(φ(a-bi)) = √(φ(a+bi)φ(a-bi))

Cor. 2. Theorem and Corollary hold for φ(c_1,c_2...,c_n): c_i∈**C**

Cor. 3. The modulus of a product of complex numbers is equal to the product of their moduli.

Cor. 4. The modulus of a quotient of complex numbers is equal to the quotient of their moduli.

Look at how the form of one thing leads unexpectedly to the form of another thing:

$(x_1+y_1i)(x_2+y_2i) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$

∴ norm(LHS) = norm(RHS)

$= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2$ (by def.)

$= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2$

$= x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)$

$= (x_1^2 + y_1^2)(x_2^2 + y_2^2)$

Now let x_i,y_i above be ∈**N**

∴ ∀product of two integers, each a sum of perfect^{2}s, is a sum of two perfect^{2}s.

Sym. (Do the algebra) A product of three such integers is a sum of two perfect^{2}s.

By induction, ∏(n such integers) is a sum of two perfect^{2}s.

So complex numbers lead to a result in number theory.

Thm. 3.9. The modulus of a sum of complex numbers is always less than or equal to the sum of their moduli.

Proof

Using the graphical representation of complex numbers in DME, we know that only sums of parallel or concurrent vectors can be equal to the sum of their lengths. So in most cases, the modulus of the sum will be less than the sum of the moduli. ■

Algebraic Functions with Variables in C

Recall that rational fns only use the four operators +, -, ×, ÷. Algebraic fns add the extraction of roots or the ^{n}√ operator with n∈**N**. We want to confirm that the values of such fns with vars ∈ **C** produce all their results in **C**. So we need to show that the nth root of ∀c∈**C** is itself in **C**. Consider the case of the square root:

√(x + yi) = X + Yi

x + yi = $X^2 - Y^2$ + 2XYi : X,Y∈**R**

∴ $X^2 - Y^2 = x$ 2XY = y

∴ $(X^2 + Y^2)^2 = x^2 + y^2$

∴ $X^2 + Y^2$ = +√($x^2 + y^2$)

∴ $2X^2$ = +√(x^2+y^2) + x

∴ X^2 = (+√(x^2+y^2) + x)/2

∴ X = ±√((+√(x^2+y^2) + x)/2)

Sym. $Y = \pm\sqrt{((+\sqrt{(x^2+y^2)} - x)/2)}$
$x^2=y^2 > x$ ∴ quantities in radicals are real
Because 2XY = y, we take unlike signs as y is pos. or neg.
y pos. ⇒ $\sqrt{((x+yi)} = \pm(\sqrt{((+\sqrt{(x^2+y^2)} + x)/2)} + i(\sqrt{((+\sqrt{(x^2+y^2)} - x)/2)}))$
y neg. ⇒ use -i above instead of i
Note that the values of X,Y are in **Q** only when $x^2+y^2 = q^2$ for some q∈**Q**.

Example

Express √i and √-1 as complex numbers.
$\sqrt{i} = x + yi$ [1]
$i = x^2 - y^2 + 2xyi$
∴ $x^2-y^2 = 0$ $2xy = 1$
∴ $(x+y)(x-y) = 0$ ∴ y = x ∨ -x but -x↴ (2xy = 1) ∴ y = x
∴ $2x^2 = 1$ ∴ $x = \pm 1/\sqrt{2}$
∴ $\sqrt{i} = \pm(1 + i)/\sqrt{2}$ (from [1])
Sym. $\sqrt{-i} = \pm(1 - i)/\sqrt{2}$

Thm. 3.10. ∀nth root r of ∀c∈**C** ⇒ r∈**C**.
1) ∀c∈C, ∃n roots of c in C
General case of the nth root of r(cosθ + isinθ) ∈ **C**
r > 0 ∴ $\exists! r^{1/n} > 0$
Consider n numbers in **C**:

$r^{1/n}(\cos(\theta/n) + i\sin(\theta/n))$ [1]
$r^{1/n}(\cos((2\pi+\theta)/n) + i\sin((2\pi+\theta)/n))$ [2]
$r^{1/n}(\cos((4\pi+\theta)/n) + i\sin((4\pi+\theta)/n))$ [3]
…
$r^{1/n}(\cos((2s\pi+\theta)/n) + i\sin((2s\pi+\theta)/n))$ [s+1]
…
$r^{1/n}(\cos((2(n+1)\pi+\theta)/n) + i\sin((2(n+1)\pi+\theta)/n))$ [n]

All distinct as all args < 2π. Each to the nth power = r(cosθ + isinθ)
i.e. $(s+1)^n = (r^{1/n})^n(\cos((2s\pi+\theta)/n) + i\sin((2s\pi+\theta)/n))^n$
$= r(\cos(n(2s\pi+\theta)/n) + i\sin(n(2s\pi+\theta)/n))$ (De Moivre's Thm)
$= r(\cos(2s\pi+\theta) + i\sin(2s\pi+\theta)$
$= r(\cos\theta + i\sin\theta)$
2) ∀n∈N,c∈C, ∃n nth roots and only n nth roots
Let a = r(cosθ + isinθ), z = ∀nth root of a ∴ $z^n = a$ ∴ $z^n - a = 0$
∀fn φz n° ∃n roots ∴ a, $z-z_1$, $z-z_2$, $z-z_3$, …, $z-z_{n-1}$ are all the factors and are the only factors. ■
Cor. 1. **R** ⊂ **C** ∴ ∀r∈**R**, ∃n nth roots of r.
Cor. 2. Imaginary nth roots of r∈**R** exist in conjugate pairs. Because if r real ⇒ if $(x+yi)^n = r$ then $(x-yi)^n = r$.

It follows that ∀r∈$\mathbf{R}^+$ (or [0,+∞)) r has form r(cos0 + i sin0) and ∀r∈$\mathbf{R}^-$ ((-∞,0]) has form r(cosπ + i sinπ).
∴ nth roots of ±1 give us the nth roots for ∀r∈**R**.

And we worked out the nth roots of positive and negative unity in DME. The first of the imaginary roots of unity:

$$\cos(2\pi/n) + i\sin(2\pi/n) \qquad [1]$$

is **primitive**, which is any nth root which is not also a root of an order less than n.
OR $r = \cos 2\pi/3 + i\sin 2\pi/3$ is the 6th root of +1 but also a cube root of +1. So it is not primitive. Depending on n, there can be multiple primitive nth roots. Denote [1] as ω.

$\omega^s = (\cos(2\pi/n) + i\sin(2\pi/n))^s = \cos(2s\pi/n) + i\sin(2s\pi/n)$ by De Moivre's Thm
$\omega^n = (\cos(2\pi/n) + i\sin(2\pi/n))^n = \cos 2\pi + i\sin 2\pi = 1$
∴ our ω is a primitive imaginary nth root of unity and roots are $\omega, \omega^2, \omega^3, \ldots, \omega^n$.

Sym. ω' is a primitive nth root of -1 and the nth roots of -1 are $\omega'^1, \omega'^3, \omega'^5, \ldots, \omega'^{2n-1}$.
So the $r(\cos\theta + i\sin\theta)$ form of ω' is what? It follows:

Thm. 3.11. ∀binomial ifn form $(x^n \pm A)$ has n factors with coeffs ∈ **C** or can be factored into at most two real factors 1° and remaining real factors of 2°:

Example

1) $x^{2m} - a^{2m}$ vanishes for any $\omega^i a$ [1-2m]
$\therefore = (x - a\omega)(x - a\omega^2)\cdots(x - a\omega^{2m})$
To get real factors we first use $(x + a)(x - a)$
then we have roots $a(\cos(2s\pi/n) + i\sin(2s\pi/n)$ or
$(x - a\cdot\cos(2s\pi/n) - a\cdot i\cdot\sin(2s\pi/n))(x - a\cdot\cos(2s\pi/n) + a\cdot i\cdot\sin(2s\pi/n))$
$= (x - a\cdot\cos(2s\pi/n))^2 + a^2\sin^2(2s\pi/n)$
$= x^2 - 2ax\cdot\cos(2s\pi/n) + a^2$
So the factors are:
$(x + a)(x - a)(x^2 - 2ax\cdot\cos(2\pi/n) + a^2)(x^2 - 2ax\cdot\cos(4\pi/n) + a^2)\cdots$
where the quadratics may or may not have real factors.

2) ω imaginary cube root of unity.
Then 1) $1 + \omega + \omega^2 = 0$ and 2) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) \in \mathbf{R}$.
1) $1 + \omega + \omega^2 = (1 - \omega^3)/(1 - \omega) = 0$ as $\omega^3 = 1$
2) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = \omega^3 x^3 + (\omega^4 + \omega^2)xy + \omega^3 y^2$
$\omega^3 = 1 \quad \omega^4 + \omega^2 = \omega + \omega^2 = -1$ as $1 + \omega + \omega^2 = 0$
$\therefore (\omega x + \omega^2 y)(\omega^2 x + \omega y) = x^2 + xy + y^2 \in \mathbf{R}$

Fundamental Theorem of Algebra ∀ifn of form [ƒ] n°, coeffs ∈ **C** has n roots.

The Fundamental Theorem of Arithmetic is that any whole number has unique prime factors. This theorem says the same thing about polynomials. The best proof uses Gauss's Theorem:

Gauss's Theorem ∀alg.eqn f of form [ƒ] n° with coeffs ∈ **C** ⇒ ∃a = c+di: f(a) = 0.

Gauss gives us one root. Remainder Theorem makes that root a factor leaving another f of (n-1)°, which has Gauss's one root, its factor, and an f of (n - 2)°, ... or wash, rinse, repeat. Chrystal's proof of Gauss's Theorem is the ugliest thing in his book so far and three pages long. If you care, go find a version of Gauss's Theorem that you like. But we uniquely factor whole numbers without having to prove anything. So we can trust Gauss and factor away. Our ifns are "as integers". Mostly, we don't need to know every proof. Decide whether or not you need to know this one and find a prettier one than Chrystal's. You will see later (to my immense satisfaction) that Murphy simply assumes Gauss's theorem when he talks about this.

Another way to look at the quadratic real factors above is that if a+bi is a root then a-bi is a root and $(x - a + bi)(x - a - bi) = (x - a)^2 + b^2$. So this factor is fully in **R**. So one take away here is that any real fn (rfn) has real factors of positive integral powers of degrees one or two.

Equations of Condition

Equations of condition are not identities. They are one or more equations solved by one or more set of values assigned to the variables. If only one set, equations are **singly determinate**. If more than one set, **multiply determinate**. If no solution, **indeterminate**. The following propositions apply to equations of condition.

Prop. 3.1. The soln of a system of eqns is in general determined when the number of eqns equals the number of vars.

Prop. 3.2. If the number of eqns is less than the number of vars the soln is generally **indeterminate**, i.e. infinite solns exist.

Prop. 3.3. If the number of eqns is greater than the number of vars there is in general no soln and the system is **inconsistent**.

Prop. 3.4. An ifn of n° in one var has n roots, real or complex, single or multiple.

Prop. 3.5. A determinate system of integral (ifn) equations of m vars whose degrees in those vars are p,q,r,... has at most $\prod pqr\cdots$ solns and generally exactly that many solns.
Cor. 1. If more solns than $\prod pqr\cdots$ are found, the system is indeterminate with infinite solns.

Def. Two systems of equations A,B are **equivalent** when every soln of one is a soln of the other. If in deriving B from A, each step maintains equivalence, each step is **reversible**. If a step destroys equivalence, that step is **irreversible**.

Let P,Q be two fns in (x,y,z,...) such that for any value of the vars, P,Q are finite. If $P \cdot Q = 0$ and $Q \neq 0$ then $P = 0$. And if $P \cdot Q = 0$ and $P \neq 0$ then $Q = 0$. Otherwise, the only values that make $P \cdot Q = 0$ are the roots of P or Q. It follows that if $P = Q$ then $P \pm R = Q \pm R$ where R is either constant or is any fn of the vars.

So just as with integers

$$P + Q = R + S \Rightarrow P + Q - S = R$$

and similar, where S is either a constant or a fn of the vars. Therefore, any system can be reduced to the form R = 0:

$$P = Q \Rightarrow P - Q = Q - Q \Rightarrow P - Q = R = 0$$

For multiplication, $P = Q \Rightarrow PR = QR$ if R is a non-zero constant but not if R is a fn of the vars because the following

$$PR = QR \Rightarrow PR - QR = 0 \Rightarrow (P - Q)R = 0$$

is satisfied by values that make R = 0 which will not satisfy P - Q = 0. Dividing both sides of an equation by anything other than a non-zero constant is irreversible as we may lose solns.

Example

$(x - 1)x^2 = 4(x - 1)$ [1]
$x^2 = 4$ $(\div (x-1))$
$\therefore x^2 - 4 = 0 \;\therefore\; (x + 2)(x - 2) = 0$ solns: 2,-2
But $[1] = (x - 1)(x + 2)(x - 2)$ solns: 1,2,-2

Since multiplying or dividing by a constant is reversible, if we have an eqn with coeffs $\in \mathbf{Q}$, we can find an equivalent eqn with coeffs $\in \mathbf{Z}$ by using LCM and we can take any ifn of form [ƒ] and change c_0 to unity.

Example

$$((p+q)/q\cdot x + p/(p - q)\cdot y)((p - q)/p\cdot x + q/(p+q)\cdot y) = 2xy$$

Multiply both sides by $p\cdot q\cdot(p+q)\cdot(p - q)$ and it resolves to:

$$(p^2- q^2)x^2 + p^2q^2y^2 = 0$$

Every ifrac is reducible to an ifn but this may introduce extraneous solns.

$2x - 3 + (x^2- 6x + 8)/(x - 2) = (x - 2)/(x - 3)$ [1]
$(2x - 3)(x - 2)(x - 3) + (x^2- 6x + 8)(x - 3) = (x - 2)^2$

This is an ifn and solved by any soln of [1]. But do any solns of (x - 2)(x - 3) solve [1]? Clearly, x = 3 does not. But x = 2 does. Now $x^2- 6x + 8 = (x - 2)(x - 4)$. Therefore [1] is 2x - 3 + x - 4 = (x - 2)/(x - 3) and x = 2 is not a soln of this equation. We have introduced it falsely and must discard it.

$P = Q \Rightarrow P - Q = 0$
multiply by $(P^{n-1}+ P^{n-2}Q + \cdots + Q^n) \Rightarrow P^n - Q^n = 0$

But here we introduce solns. Sym. rationalizing factors introduce solns at every step. It can be even worse:

$$\sqrt{(x + 1)} + \sqrt{(x - 1)} = 1 \qquad [1]$$
$$\sqrt{(x + 1)} = 1 - \sqrt{(x - 1)}$$
$$x + 1 = 1 + x - 1 - 2\sqrt{(x - 1)}$$
$$1 = -2\sqrt{(x - 1)}$$
$$1 = 4(x - 1)$$
$$4x - 5 = 0$$
$\therefore x = 5/4$ which is **not** a soln of [1]

Given system A: $P_i = 0$ [1 - n] if we derive system B:

$$L_1P_1 + L_2P_2 + \cdots + L_nP_n = 0 \quad P_2 = 0 \quad \ldots \quad P_n = 0$$

the two are equivalent if L_1 is a non-zero constant. And if $l, l', m, m' \in \mathbf{R}$ any of which can be zero but $lm' - l'm \neq 0$ then

A: $U = 0 \quad U' = 0$
B: $lU + l'U' = 0 \quad mU + m'U' = 0$

are equivalent systems. And given

A: $P = Q \quad R = S$

any soln of A is a soln of

B: $PR = QS \quad R = S$

but the systems are not equivalent. Any soln of A is a soln of B. But B is solved by solns of $R = 0$ and $S = 0$ which are not in A. An important method of soln is elimination. Say you have two equations in x,y. You eliminate y in one of them. The other retains your relation of y to x. A simple example of this would be

A: $x^2 + y^2 = 1 \qquad x + y = 1$
We eliminate y from the first eqn.
$x + y = 1 \therefore y = 1 - x$
$\therefore x^2 + (1 - x)^2 = 1$
$x^2 + 1 - 2x + x^2 = 1$
$x^2 - x = 0$
$\therefore$ B: $x(x - 1) = 0 \qquad x + y = 1$ is an equivalent system to A

which seems trivial here. But elimination is very useful when the systems get larger and more complicated.

Examples

1) Rationalize $\sqrt{X} \pm \sqrt{Y} \pm \sqrt{Z} \pm \sqrt{U} = 0$
We get a general soln by taking only + as squaring reduces then all to the same thing.

$\sqrt{X} + \sqrt{Y} = -\sqrt{Z} - \sqrt{U}$ (square)
$X + Y + 2\sqrt{XY} = Z + U + 2\sqrt{ZU}$
$X + Y - Z - U = 2\sqrt{ZU} - 2\sqrt{XY}$ (square)
$(X + Y - Z - U)^2 = 4XY + 4ZU - 8\sqrt{XYZU}$
$\sum X^2 - 2\sum XY = -8\sqrt{XYZU}$
$(\sum X^2 - 2\sum XY)^2 = 64XYZU$

But this general soln is not the simplest method:

$\sqrt{(2x+3)} + \sqrt{(3x+12)} - \sqrt{(2x+5)} - \sqrt{3x} = 0$
$\sqrt{(2x+3)} + \sqrt{(3x+12)} = \sqrt{(2x+5)} - \sqrt{3x}$
$5x + 5 + 2\sqrt{(6x^2+ 13x + 6)} = 5x + 5 + 2\sqrt{(6x^2 + 15x)}$
$6x^2+ 13x + 6 = 6x^2 + 15x$
$6 = 2x$
$3 = x$

2) If $x + y + z = 0$ [1]
then $\sum(y^2+ yz + z^2)^3 = 3\prod(y^2+ yz + z^2)^3$
$y^2+ yz + z^2 = y^2+ z(y + z)$
$= (-z - x)^2 + z(-x)$ (by [1])
$= z^2+ zx + x^2$
$= x^2+ xy + y^2$ (by sym.)
$\therefore \sum(y^2+ yz + z^2)^3 = 3(y^2+ yz + z^2)^3$ and $3\prod(y^2+ yz + z^2) = 3(y^2+ yz + z^2)^3$

3) Given $x,y,z \neq 0$, eliminate x,y,z in [1]-[3]
$y^2+ z^2 = ayz$ [1]
$z^2+ x^2 = bzx$ [2]
$x^2+ y^2 = cxy$ [3]
$x^2- y^2 = (bx-ay)z$ ([2]-[1]) [4]
$\therefore bx-ay \neq 0$. Else $x^2 = y^2 \therefore$ By [3] $x = 0$↴
$z^2(bx-ay)^2+ x^2(bx-ay)^2 = bzx(bx-ay)^2$ ($[2]\cdot(bx-ay)^2$)
$(x^2- y^2)^2 + x^2(bx-ay)^2 = bx(bx-ay)(x^2- y^2)$ ([4])
$\therefore (x^2- y^2)^2 = xy(ax-by)(bx-ay)$
$\therefore (x^2+ y^2)^2 - 4x^2y^2 =$ RHS [5]
$(c^2- 4)xy = ab(x^2+ y^2) - (a^2+ b^2)xy$ (from [3], [5] becomes this)
$\therefore (a^2+ b^2+ c^2 - 4)xy = ab(x^2+ y^2)$ [6]
$\therefore (a^2+ b^2+ c^2 - 4 - abc)xy = 0$ (from [3])
$xy \neq 0 \therefore (a^2+ b^2+ c^2 - 4 - abc) = 0$ [7]
This [7] is the **eliminant** or **resultant** of this system of equations.

Functions

A few remarks on algebraic fns of form [ƒ] with coeffs ∈ **R**.

1) f(x) can be infinite for some finite x, usually when approaching form 1/0.
2) f(x) can instantaneously increase or decrease but only by definition as in
f(x) = 3x + 1 ∈ (-∞,1), 3x + 2 ∈ [1,∞). Otherwise, alg. fns are continuous.

Def. A fn f(x) is **continuous** if f(x) has a limit b if as x→a then f(a) = b. Or

limit f(x) as x→a = f(a)

Intuitively, this means you can graph any interval of f(x) without lifting your pencil from the paper. Or, you always get to where you are going. No surprises.

3) If c_i∈**R**, f(x) can be complex for some real values of x. Let f(x) = +√(1 - x^2) then f(x) has real values only on [-1,1] and imaginary values elsewhere. But this cannot happen if f(x) is a rational algebraic fn using only the four operations of arithmetic and no radicals.

These next six propositions refine some ideas in DME and can be proved using the ideas of limits from that text. They consider what happens on an interval where the two fns approach extreme values. Keep in mind that there could be many intervals where this happens if we consider the whole domain of alg. fns P,Q.

Prop. 3.5. P finite, Q→0 ⇒ PQ→0

Prop. 3.6. P finite, Q→∞ ⇒ PQ→∞

Prop. 3.7. P!→0 when Q→0 ⇒ (Q→0 ⇒ P/Q →∞)
If P∧Q→0, the form becomes 0/0 which is indeterminate.

Prop. 3.8. P!→∞ when Q→∞ ⇒ (Q→∞ ⇒ P/Q →0)

Prop. 3.9. P∧Q →0 ⇒ P+Q →0

Prop. 3.10. (P∨Q → ±∞) ∨ (P∧Q → (+∞ ∨ -∞)) ⇒ P+Q ⇒ ±∞

We can extend these ideas:

Prop. 3.11. If P = ∏P_i [1-n] then P remains finite if all P_i remain finite.
If any P_i→0 then P→0 and if any P_i→∞ then P→∞.

Prop. 3.12. If S = ∑P_i [1-n] then S remains finite if all P_i remain finite.
S→0 only if all P_i → 0. But S→∞ if any P_i →∞.

Along these same lines, consider P/Q:

Prop. 3.13. P/Q finite if P and Q finite.
It may be finite is $P \wedge Q \to 0$ or $P \wedge Q \to \infty$.

Prop. 3.14. $P/Q = 0$ if $(P=0 \wedge Q \neq 0) \vee (P \neq \infty \wedge Q = \infty)$
It may equal zero if $P \wedge Q \to 0$ or $P \wedge Q \to \infty$.

Prop. 3.15. $P/Q = \infty$ if $(P=\infty \wedge Q \neq \infty) \vee (P \neq 0$ and $Q=0)$
It may equal go to infinity if $P \wedge Q \to 0$ or $P \wedge Q \to \infty$.

The above ideas have implications regarding continuity.

Prop 3.16. The algebraic sum of a finite number of continuous fns is a continuous fn.

Prop 3.17. The product of a finite number of continuous fns is a continuous fn so long as all the factors remain finite.
Cor. 1. If A constant, P continuous fn ⇒ AP continuous fn.
Cor. 2. If A constant, $m \in \mathbf{N} \Rightarrow Ax^m$ continuous fn.
Cor. 3. ∀ifn f(x) is continuous and cannot become infinite for a finite value of x.

Prop. 3.18. If P,Q ifns of x then P/Q is finite and continuous for all finite values of x where $Q(x) \neq 0$.

The underlying idea in these last three propositions is that in each case nothing occurs to take f(a) away from the limit of f(x) as $x \to a$. In addition or subtraction, the increments of the elements are all continuous, so the sum of the increments is continuous. Nothing breaks the path to the limit. Then in multiplication, we can't let any factor go to infinity or it produces a discontinuity, And the same happens in division if the denominator runs off to zero.

If $P = A+B$, $P(x+h) = A+a + B+b = A+B + a+b$. So the increment of the fn is a+b. The limits of A,B being continuous, the limit of the sum is continuous, or rather, the sum provides continuity. In a product $(A+a)(B+b)$, we have $AB + Ab + Ba + ab$ which goes to AB as $a,b \to 0$ and continuity is preserved so long as A and B are well-behaved and remain finite. You can work out the symmetric ideas for quotients on your own.

Thm. 3.12. If f(x) continuous on [a,b] where f(a) = p, f(b) = q, then f(x) passes at least once through every value on [p,q].
Cor. 1. If f(a),f(b) different signs then f(x) has at least one root, and in any case an odd number of roots, on [a,b].
Cor. 2. If f(a),f(b) same sign, if f(x) has any roots on [a,b] there must be an even number of them. Note that if f(x) merely touches line ab, the roots are multiple roots.
Cor. 3. A continuous fn can change sign only by passing through the value zero.
Cor. 4. If P,Q continuous ifns and p(P,Q) (or "P,Q prime to e.o.") then P/Q can only change sign by passing through 0 or ∞.

Arguably, these are all provable by graphing any continuous fn. Note that the Remainder Theorem allows you to exclude the case of 0/0 and therefore Cor. 4. follows from Cor. 3. This next theorem strikes me as a variation of De Morgan's theorem that by choosing x, any term of a fn can contain all the following terms.

Thm. 3.13. ∀ifn f(x): 1) By taking x small enough f(x) can have the same sign as its lowest term and 2) by taking x large enough, f(x) can have the same sign as its largest term.

Proof

Take $y = px^3 + qx^2 + rx + s$

1) If s≠0 then by inspection we can make $px^3 + qx^2 + rx < s$

If s=0, we have $(px^2 + qx + r)x$ and the same reasoning holds.

2) We can alter the fn to

$y = x^3(p + q/x + r/x^2 + s/x^3)$

where for large x we make $y = px^3$. ■

Thm. 3.14. ∀ ifn f(x) of form [ƒ] can be made a rifn with $c_0=1$ ∴

if f odd° then f(+∞) = +∞ and f(-∞) = -∞

if f even° then f(±∞) = +∞

Cor. 1. ∀ifn odd°, coeff ∈ **R** has at least one real root and if more then an odd number.

Cor. 2. If ∀ifn even° has real roots, it has an even number of them

Cor. 3. ∀ifn, coeff ∈ **R**, if it has complex roots, it has an even number of them.

Maxima and Minima

Chrystal introduces here the ideas of maxima and minima. But without the Calculus, these require lots of hand-waving. Fortunately from DME we have enough Calculus for this.

Def. f(x) has a **maximum** where f'(x) (first derivative of f(x)) changes from positive to negative. The curve stops going up and starts coming down at this point. And f(x) has a **minimum** where f'(x) changes from negative to positive. Therefore wherever f(x) has a maximum or minimum, f'(x) = 0.

A couple of caveats. A fn f(x) could go up, decrease its tangent to zero and then go up again, making a kind of flattened S-curve. And f'(x) would equal zero here too. So in every case you have to test f(x) on both sides of f'(x)=0. Also, f(x) could have a maximum or minimum where, by having some kind of instantaneous change, it would have no f'(x).

So to find maxima and minima for f(x):

1. derive f'(x);
2. solve for f'(x) = 0; and
3. test roots for maxima, minima, and false positives.

Example

$y = x^3/3 - 2x^2 + 3x + 2$

$y' = x^2 - 4x + 3$ roots: 1,3

So we have a max or min at both (1,3⅓) and (3,2) and you can determine which is which.

Thm. 3.15. If f(x) is continuous, its maxima and minima succeed e.o. alternately.

Thm. 3.16. If f(x) not constant and has real roots a,b: a<b then there must be at least one max or min on [a,b].

Thm. 3.17. Let x=p be a max or min of f(x) then the fn f(x)-p has a double root.
(Use a quick sketch of some f(x) to see that this theorem is saying.)

Functions of Two Variables

Let z = f(x,y) be an ifn, x,y,coeff ∈ **R**. By considering only such z, the value of z will be real. Our increment of f is now f(x+h,y+k) - f(x,y). But with these terms in form $Ax^m y^n$ you could easily prove that as x goes from a→a' and y from b→b' that z must pass continuously from some c→c'.

Now let P = (a,b), P' = (a',b'). There are an infinity of continuous curves from P to P'. Let line PQ have the magnitude of f(x,y) at P where Q is above the XY plane if positive, below if negative. Then the locus of all possible Q is a graphical surface of f(x,y). As P travels any curve S in the XY plane, Q travels curve ∑ on the surface of f(x,y) so that S is the orthogonal projection of ∑.

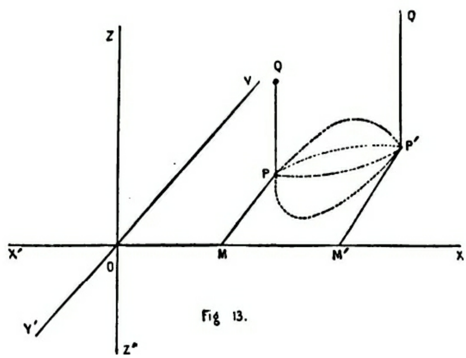


Fig 13.

If we want to know where f(x,y)=c, we cut the surface with some plane U || plane XY at distance c above or below the XY plane as c is positive or negative. Plane U intersects the surface at some curve ∑ which is the **contour line** of c on the surface. If we let U = XY, this contour is f(x,y) = 0. This is the boundary, on either side of which f(x,y) has opposite signs.

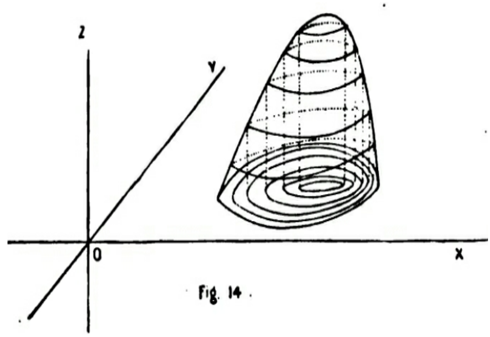


Fig. 14.

So we have analogs of the above theorems, such as

Thm. 3.18. If P = f(a,b) > 0 and Q = f(a',b') < 0 then if f continuous f passes through zero an odd number of times as it passes from P to Q.

And in general it follows that any plane curve can be analytically represented by a fn of two variables. This is only a preview of two var fns. A similar preview of complex fns follows.

Complex Functions of a Single Variable

Let f(x) be an ifn of x,coeffs ∈ **C**. Here x = a + bi. And f(a+bi) is continuous from a+bi to a'+b'i if a varies continuously from a to a' and b from b to b'. We know that any fn f(a+bi) reduces to some A + Bi where A,B ifn with coeffs ∈ **R**. So f(a+bi) is continuous if A + Bi is continuous. Graphically, we can say that the graph of the ind var is a continuous curve S when the graph of the dependent var (f(a+bi)) is a continuous curve S'.

We can think of A,B in A + Bi as φ(a,b), ψ(a,b). So the values that make A = 0 will make φ = 0. Sym. for B and ψ. Where φ=0 we have a curve S and where ψ=0, a curve T. Therefore the intersection S∩T is the roots of f(a+bi). Which I find aesthetically very pleasing.

Example

$y = ix^2 + 8$

$A + Bi = i(a+bi)^2 + 8$

$= 2(4 - ab) + (a^2 - b^2)i$

$\therefore A = 2(4 - ab) \quad B = a^2 - b^2$

Curves S and T are derived from making b a fn of a or b(a)

S) b = 4/a

T) b = +a,-a

∴ S is an hyperbola intersected by the line b = a at P = (2,2) and Q = (-2,-2)

∴ roots of $ix^2 + 8$ are 2 + 2i and -2 - 2i.

Equations 1°

Let's look at a few things we already know but now from Chrystal's viewpoint.

$$ax + b = 0$$
$$\therefore a(x - (-b/a)) = 0$$
$$\therefore \text{soln: } x = -b/a$$

Thm. 3.19. ∀eqn 1° has unique soln.

Proof

Let there two solns: α,β and a≠0

$a\alpha + b = 0$

$a\beta + b = 0$

$a(\alpha - \beta) = 0 \qquad (-)$

$a \neq 0 \;\therefore\; \alpha - \beta = 0 \;\therefore\; \alpha = \beta$ ■

Def. A system (sys) of eqns is **consistent** if a unique soln satisfies all eqns in the sys.

Two eqns 1° are generally inconsistent. Given ax + b = 0 and cx + d = 0, consistency requires ad - bc = 0 where, geometrically speaking, lines are parallel (||). If b,d = 0, eqns are consistent as soln is x = 0. And if two are consistent, then for some r∈**R**:

$$ax + b = r(cx + d)$$

∀eqn 1° of two vars has infinite solns. The form is ax + by + c = 0. Given any y,

$$x = -(by + c)/a$$

so we can think of y here as an **arbitrary constant** and the eqn as having a **one-fold infinity** of solns.

∀l,l',m,m' ∈ **Q**: lm' - l'm ≠ 0, then these two systems (1,2 and 3,4) are equivalent.

$$ax + by + c = 0 \quad [1]$$
$$a'x + b'y + c' = 0 \quad [2]$$
$$l(ax + by + c) + l'(a'x + b'y + c') = 0 \quad [3]$$
$$m(ax + by + c) + m'(a'x + b'y + c') = 0 \quad [4]$$

Clearly, the two systems have only the same solns and solns of [3],[4] give:

$$m'(l(ax + by + c) + l'(a'x + b'y + c')) - l'(m(ax + by + c) + m'(a'x + b'y + c')) = 0 \quad [5]$$
$$-m(l(ax + by + c) + l'(a'x + b'y + c')) + l(m(ax + by + c) + m'(a'x + b'y + c')) = 0 \quad [6]$$

And [5],[6] reduce to:

$$(lm' - l'm)(ax + by + c) = 0 \quad [7]$$
$$(lm' - l'm)(a'x + b'y + c') = 0$$

which if lm' - l'm ≠0 are equivalent to [1] and [2].

If we let l = +b', l' = -b, m = -a', m = +a then lm' - l'm = ab' - a'b and [3],[4] become:

$$(ab' - a'b)x + cb' - c'b = 0 \quad [3']$$
$$(ab' - a'b)y + c'a - ca' = 0 \quad [4']$$

$$\therefore\ x = \frac{cb' - c'b}{ab' - a'b} \qquad y = \frac{ac' - a'c}{ab' - a'b}$$

which is unique if ab' - a'b ≠ 0. Eqns [1],[2] are collaterally symmetrical wrt xy, ab, cd. So if we know x then y is obtained by subbing b for a, a for b, a' for b' and b' for a'.

Bezout's Method

If we start with [1],[2] and any λ, any soln of the sys solves

$$(ax + by + c) + \lambda(a'x + b'y + c') = 0 \qquad [3]$$

or

$$(a + \lambda a')x + (b + \lambda b')y + (c + \lambda c') = 0 \qquad [4]$$

Select any λ, let $y = 0$ and $[4] \rightarrow (a + \lambda a') + (c + \lambda c') = 0$
But $\lambda = -b/b'$ $\therefore$ $x = (c + \lambda c')/(a + \lambda a') = (b'c - bc')/(ab' - a'b)$ Sym. for y.

If we begin with sys [1],[2], this is an equivalent sys:

$$y = (-ax + c)/b$$
$$a'x + b'y + c' = 0$$

and subbing, this is an equivalent sys:

$$y = (-ax + c)/b$$
$$a'x - (b'(ax + c))/b + c' = 0$$

And algebrating, we get the same results. So our original sys [1],[2] is consistent if $ab'-a'b \neq 0$ or, from DME, the determinant of the coeffs is non-zero. If we add the eqn, $a''x + b''y + c'' = 0$ to our sys, the sys has a soln if the determinant

$$\text{Det} \begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix} \neq 0 \quad \text{OR} \quad a''(bc'-b'c) + b''(ca'-c'a) + c''(ab'-a'b) \neq 0 \qquad [1]$$

For **linear** eqns, a non-zero n×n determinant is the condition of consistency for any sys of n eqns. The determinant [1] shows the consistency of three eqns in either of these forms:

$$ax + by + c = 0$$
$$ax = by + cz + d = 0$$

Now all of what we have said here scales up with linear equations. Any eqn 1° of three vars $ax + by + cz + d = 0$ has a **two-fold infinity** of solutions. Assign any arbitrary constants for y and z and x is then determined. A sys of two such eqns has a one-fold infinity of solns:

$$ax + by + cz + d = 0$$
$$a'x + b'y + c'z + d' = 0$$

Choose any z and you have the soln of two linear eqns in two vars as above. A later example shows how this can be done with an arbitrary constant and ratios.

Given a sys of three such eqns:

$$ax + by + cz + d = 0 \qquad [1]$$
$$a'x + b'y + c'z + d' = 0 \qquad [2]$$
$$a''x + b''y + c''z + d'' = 0 \qquad [3]$$

with a non-zero determinant:

$$a(b'c'' - b''c') + b(c'a'' - c''a') + c(a'b'' - a''b') \neq 0 \qquad [4]$$

But hold that thought while I make two remarks on determinants. First, these last two determinants above are the same. You can use any row for the coeffs and the others for the two-part factor. Second, if you were to take all the two-part bits in the same direction, they would alternate pos/neg and second term above would be

$$-b(a'c'' - c'a'')$$

and different texts do this in different but valid ways. Returning to our sys, it is equivalent to:

$$ax + by + cz + d = 0 \qquad [5]$$
$$c'(ax + by + cz + d) - c(a'x + b'y + c'z + d') = 0 \qquad [6]$$
$$c''(ax + by + cz + d) - c(a''x + b''y + c''z + d'') = 0 \qquad [7]$$

Then let: $A = ac' - a'c$ $\quad B = bc' - b'c$ $\quad C = dc' - d'c$

$A' = ac'' - a''c$ $\quad B' = bc'' - b''c$ $\quad C' = dc'' - d''c$

And this becomes:

$$ax + by + cz + d = 0 \qquad [5']$$
$$Ax + By + C = 0 \qquad [6']$$
$$A'x + B'y + C' = 0 \qquad [7']$$

And the determinant becomes: $AB' - A'B \neq 0$ $\qquad$ [8]

$$\therefore \qquad x = \frac{BC' - B'C}{AB' - A'B} \qquad y = \frac{CA' - C'A}{AB' - A'B}$$

Subbing these into [5] we get

$$z = \frac{-a(BC' - BC) + b(CA' - C'A) + d(AB' - A'B)}{c(AB' - A'B)}$$

We can see from geometry, that if with two eqns they are in form $ax + by = 0$ or with three eqns they are in form $ax + by + cz = 0$, then we have lines or planes which have not been displaced from the origin and solns become: $x,y = 0$ or $x,y,z = 0$ so long as the determinant (det) is non-zero.

If we have ax + by = 0 and a'x + b'y = 0 and det = 0, we then have

a(x/y) + b = 0
a'(x/y) + b' = 0

and then for any x,y: x:y::-b:a or x:y::-b':a' is a soln. Note that here these eqns are consistent. So we have a one-fold infinity of solns. And going up one dimension, if we have det = 0 with three eqns of form ax + by + cz = 0:

a(x/z) + b(y/z) + c = 0
a'(x/z) + b'(y/z) + c' = 0
a"(x/z) + b"(y/z) + c" = 0

then again, the det = 0 means the eqns are consistent and any two of them determine the ratios x:z, y:z. Therefore x : y : z = bc'-b'c : ca' - c'a : ab' - a'b where bc' can be bc', bc", or b'c" and so on sym. altering the rest of the terms depending upon which two eqns determine the ratios. So when a sys of eqns is homogeneous and consistent, the values of the vars are indeterminate **but their ratios are determinate.** If we have:

3x + 5y - 7z - 2 = 0
4x + 8y - 14z + 3 = 0
3x + 6y - 8z - 3 = 0

the simplest method of soln is our matrix method from DME which I will give in running-style for you to puzzle out:

3	5	-7	-2	0	-4	14	17	0	0	10	21	0	0	1	2.1
4	8	-14	3	1	3	-7	-5	1	0	-4	-8	1	0	0	0.4
3	6	-8	-3	0	1	-1	1	0	1	-1	1	0	1	0	3.1

∴ x = 0.4 y = 3.1 z = 2.1 Let's consider the relation of eqns to solns.

Prop. 3.19. A sys of n-r eqns 1° in n vars has a soln involving r arbitrary constants and therefore an r-fold infinity of solns.

Prop. 3.20. A sys of n eqns 1° in n vars has a unique determinate soln so long as det ≠ zero.

Prop. 3.21. A system of n+r eqns 1° in n vars is generally inconsistent. To secure consistency, r different conditions must be satisfied.

Other systems of equations can be reduced to linear systems. If all steps taken are reversible, the linear solution will be the solution of the original system. Else, solutions have been added in the reduction and results must be verified.

Example

$x^2 - y^2 = x - y$ $\quad$ $2x + 3y - 1 = 0$

LHEqn $\equiv (x - y)(x + y - 1) = 0$

$\therefore$ system equivalent to

I. $x - y = 0$ $\quad$ $2x + 3y - 1 = 0$

II. $x + y - 1 = 0$ $\quad$ $2x + 3y - 1 = 0$

Solns: I. $(1/5, 1/5)$ $\quad$ II. $(2, -1)$

Solutions of linear systems can be simplified by using **auxiliary variables**.

Examples

1) $(x - a)^3/(x + b)^3 = (x - 2a - b)/(x + a + 2b)$

Let $x+b = z$ $\therefore$ $x = z-b$ and let $c = a+b$

$\therefore$ $(z - c)^3/z^3 = (z - 2c)/(z + c)$

$(z - c)^3(z + c) = z^3(z - 2c)$

$z^4 - 2z^3c + 2zc^3 - z^4 = z^4 - 2z^3c$

$2c^3z - c^4 = 0$ $\therefore$ soln: $c/2$ $\therefore$ $x = c/2 - b = (a-b)/2$

2) $a(x+y) + b(x-y) + c = 0$ $\quad$ $a'(x+y) + b'(x-y) + c' = 0$

Let $x+y = m$ $\quad x - y = n$

$am + bn + c = 0$ $\quad$ $a'm + b'n + c' = 0$

which solves as a linear system with expressions for m,n

Then $x+y$ = expr.m $x-y$ = expr.n (expr. $\equiv$ expression of)

Add and divide by 2 for x; subtract and divide by 2 for y.

3) System of three eqns:

$x - 2y + 3z = 0$ $\quad$ [1]

$2x - 3y + 4z = 0$ $\quad$ [2]

$4x^3 + 3y^3 + z^3 - xyz = 216$ $\quad$ [3]

Rather than solving [1],[2] by choosing any z,

we can also take any arbitrary λ. Then

$x = \lambda(bc' - b'c)$ $\quad y = \lambda(ca' - c'a)$ $\quad z = (ab' - a'b)$ or

$x : y : z = bc' - b'c : ca' - c'a : ab' - a'b$

$x : y : z = 1 : 2 : 1$ or $x/1 = y/2 = z/1 = \lambda$

$\therefore x = \lambda$ $\quad y = 2\lambda$ $\quad z = \lambda$ $\quad$ [4]

$\therefore$ $27\lambda^3 = 216$ $\quad\therefore \lambda^3 = 8$

solns: $\lambda = 2$ $\quad \lambda = 2(-1 + i\sqrt{3})$ $\quad \lambda = 2(-1 - i\sqrt{3})$

Sub each of these into [4] for the actual three solns.

Graphical Remarks

A line is $y = ax + b$, therefore

1. if $b = 0$ then $y = ax$ or a line with slope a on the origin
2. if $a = 0$ then $y = b$ or a line || X-axis above or below it at distance b
3. if $a,b = 0$ then $y = 0$ or the X-axis

In #1, the root is (0,0). In #2, there is no root. In #3, $0x + 0 = 0 \therefore x = 0/0$ and line is indeterminate as all of X-axis is its "root."

A plane is $z = ax + by + c$ and its contour lines a a series of || lines. Let $k = \forall$constant, then $ax + by + c = k \;\therefore y = (-a/b)x + (k-c)/b$ which is a line. Note, however, that lines have no width, compose nothing and certainly cannot be used to build a plane. In this line the x intercept is $(k-c)/a$ and the y intercept is $(k-c)/b$. Take any two such lines and the ratio of their x and y intercepts are the same as these two lines. So all are || which is to say that the slopes all equal $-a/b$. The zero contour line here is

$$ax + by + c = 0 \qquad [1]$$

which divides the plane in two. Any point on one side of [1] makes [1] positive and any point on the other makes it negative. Two such lines, whether in the same or in different planes will have generally one point of solution. If they are parallel (see Euclid Book XI) there is no soln to the system. If they are coincident, you can derive $x = y = 0/0$ and so have an indeterminate system.

Three such lines have a soln or point of intersection iff (if and only if) det = 0. In our det, we have the cases where one or more of $ab' - a'b$, $a''b - ab''$, $a'b'' - a''b'$ vanish. Of course, if two vanish, they all vanish. So we have only two cases.

Let the lines be L, L', L".

1. Let $ab' - a'b = a''b - ab'' = 0$. Then L||L' and L||L" ∴ all lines are either parallel of coincident.
2. Let $ab' - a'b = 0$. Then L||L' and L" is neither || nor coincident with L,L'. Here the three can have no common solution unless L≡L'.

Let's prove that last one:

$$\begin{aligned}
&\text{Let } ab' - a'b = 0\\
&k = \forall\text{constant: } a'/a = b'/b = k\\
&\therefore a' = ka \quad b' = kb\\
&\therefore \det = a''(bc' - b'c) + b''(ca' - c'a) = 0\\
&\qquad = a''(bc' - kbc) + b''(cka - c'a) = 0\\
&\qquad = (a'b - ab'')(c' - kc) = 0\\
&\text{LHT} \neq 0 \text{ by hyp.} \;\therefore\; \text{RHT} = 0 \;\therefore c' = kc\\
&\therefore \text{If } k = 1 \text{ a soln exists.} \;\blacksquare
\end{aligned}$$

Equations 2°

De Morgan covered all the basic theory on quadratics. So we'll take it from there. But first, here is a

Cool Example

Solve $\sqrt{(7 + \sqrt{(7 + \sqrt{(7 + \cdots)})})}$
Let the expression equal x, then $x^2 = 7 + x$
$\therefore x^2 - x - 7 = 0$ and algebrate
This technique can be used with other infinite forms such as infinite series.

We can solve some higher degree eqns by viewing them as, or reducing them to, quadratics.

Examples

1) Solve $x^3 + 1 = 0$
   Reduces to $(x + 1)(x^2 - x + 1) = 0$
   So the three roots are all within the scope of quadratics.

2) Solve $7x^3 - 13x^2 + 3x + 3 = 0$
   Clearly divby $(x - 1)$ as $x = 1$ is a soln by inspection
   $\therefore (x - 1)(7x^2 - 6x - 3) = 0$

3) $p(ax^2 + bx + c)^2 - q(dx^2 + ex + f)^2 = 0$
   From $A^2 - B^2 = (A + B)(A - B)$
   $= (\sqrt{p}(ax^2 + \cdots) + (\sqrt{q}(dx^2 + \cdots))(\sqrt{p}(ax^2 + \cdots) - (\sqrt{q}(dx^2 + \cdots))$
   LHT mult. out is quadratic: $(a\sqrt{p} + d\sqrt{q})x^2 + \cdots$ and Sym. for RHT.

In every case, we use our understanding of the form of number to find what simple forms lie within our work. And then we exploit them. We know that we can reduce every algebraic eqn to an ifn. But we must track our work and exclude any extraneous solns we introduce.

Examples

1) $1/(x+a+b) + 1/(x-a+b) + 1/(x+a-b) + 1/(x-a-b)$
   Using $A^2 - B^2 = (A+B)(A\_B)$ combine terms 1 and 4, 2 and 3.
   $2x/(x^2 - (a+b)^2) + 2x/(x^2 - (a-b)^2) = 0$
   $\therefore 2(2x^2 - 2(a^2+b^2) = 0$
   Equiv. to sys: $x = 0 \quad x^2 - (a^2 + b^2) = 0$
   Roots: $0, \pm\sqrt{(a^2 + b^2)}$ and none introduced by process.

2) $(\sqrt{(a+x)})/(\sqrt{a} + \sqrt{(a+x)}) = (\sqrt{(a-x)})/(\sqrt{a} - \sqrt{(a-x)})$
   Rationalize denoms and algebrate:
   $4x^4 - 3a^2x^3 = 0 \therefore$ roots: $0, \pm a(\sqrt{3}/2)$
   But these are all introduced by the process and do not satisfy the original eqn.

Quadratic soln can be achieved by change of variable. If we can put an eqn in form:

$$(f(x))^2 + p(f(x)) + q = 0$$

we can solve it as a quadratic of f(x) and if f(x) is 1° or 2° then we can completely solve it with what we know so far.

Examples

1) $x^2 + 3 = 2\sqrt{(x^2 - 2x + 2)} + 2x$

$x^2 - 2x + 2 - 2\sqrt{(x^2 - 2x + 2)} + 1 = 0$

$\therefore f(x) = x^2 - 2x + 2 \;\; \therefore (f(x) - 1)^2 = 0$

$\sqrt{(x^2 - 2x + 2)} = 1$

$x^2 - 2x + 2 = 1$

$(x - 1)^2 = 0 \qquad \therefore$ roots: $x = 1$

2) $2^{2x} - 3 \cdot 2^{x+2} + 32 = 0$

$(2^x)^2 - 12(2^x) + 32 = 0$

$(2^x - 4)(2^x - 8) = 0$

Equiv to sys: $2^x = 4 \quad 2^x = 8 \;\; \therefore$ soln: $x = 2,3$

Reciprocal Equations

This is a very useful form. Consider these reciprocal eqns, i.e. where the coeffs equidistant from the ends are equal. (That was a definition, right?)

$$ax^4 + bx^3 + cx^2 + bx + c = 0$$
$$ax^4 + bx^3 + cx^2 - bx + c = 0$$

These reduce to:

$$a(x^2 + 1/x^2) + b(x + 1/x) + c = 0$$
$$a(x^2 + 1/x^2) + b(x - 1/x) + c = 0$$

Or:

$$a(x + 1/x)^2 + b(x + 1/x) + c - 2a = 0 \qquad [1]$$
$$a(x + 1/x)^2 + b(x - 1/x) + c + 2a = 0 \qquad [2]$$

Let roots of [1] = α,β [2] = γ,δ.
Then [1] is equiv to sys: $x + 1/x = \alpha \quad x + 1/x = \beta$
Or: $x^2 - \alpha x + 1 = 0 \quad x^2 - \beta x + 1 = 0$
Sym. [2] is equiv to sys: $x^2 - \delta x - 1 = 0 \quad x^2 - \gamma x - 1 = 0$
And the roots of these pairs are the roots of the 4° original eqn.

More generally, if we have:

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

it can take the form:

$$a(x^2 + e/(ax)) + b(x + d/(bx)) + c = 0$$

and if $e/a = d^2/b^2$ this takes the form:

$$a(x + d/(bx))^2 + b(x + d/(bx)) + c - 2\cdot ad/b = 0$$

Further, if we have:

$$ax^5 + bx^4 + cx^3 \pm cx^2 \pm bx \pm a = 0$$

where either the upper or lower signs are all taken together, this has form:

$$a(x^5 \pm 1) + bx(x^3 \pm 1) + cx^2(x \pm 1) = 0$$

where either (x+1) or (x-1) is a factor. And removing this factor we obtain a reciprocal 4° eqn as above.

Elimination

I do not yet fully grasp the methods of elimination. But I can see the importance of it. Let me share as much of it as I have grasped and you can dig into it more deeply.

Let's look at systems with quadratics. If we have a sys of 2 eqns l° (that's not a 1) and m° in x,y, the sys has l·m solns. If we eliminate y, the sys in x will be (l·m)°. And this eqn is the **resultant eqn** in x. The same idea extends to sys with more than two eqns and more than two vars. Generally, such a sys has (n-1) 1° eqns and one quadratic. By first solving the (n-1) 1° eqns, we substitute those values in the quadratic. This gives two root solns which in turn gives a second value to the (n-1) solns.

The idea of elimination, simplest case, is: given two eqns in x,y, we eliminate y in one of them, solve for x and sub that value back into the other eqn for y. And this basic idea scales up for more eqns and more vars.

Example

$lx + my + n = 0$ [1]
$ax^2 + 2bxy + by^2 + 2gx + 2fy + c = 0$ [2]
[1] equiv to: $y = -(lx + n)/m$ [3]
sub $y \rightarrow$ [2] $= am^2x^2 - 2hmx(lx+n) + b(lx+n) + 2gm^2x - 2fm(lx+n) + cm^2 = 0$
$= (am^2 - 2hlm + bl^2)x^2 + 2(gm^2 - hmn + bnl - flm)x + (bn^2 - 2fmn + cm^2) = 0$ [4]
∴ sys [1],[2]≡[3],[4] and [4] gives two values of x.
Sub x's into [3] for two values of y.

More generally consider:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad [1]$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \qquad [2]$$

As quadratics in y, these are:

$$by^2 + 2(hx+f)y + (ax^2 + 2gx + c) = 0$$
$$b'y^2 + 2(h'x+f')y + (a'x^2 + 2g'x + c') = 0$$

which we can abbreviate as:

$$by^2 + px + q = 0 \qquad [1']$$
$$b'x^2 + p'x + q' = 0 \qquad [2']$$

Mult. by b,b' and subtract, mult. by q' and then q and subtract:

$$(pb' - p'b)y + (b'q - bq') = 0 \qquad [3]$$
$$(b'q - bq')y^2 + (p'q - pq')y = 0 \qquad [4]$$

If $b'q - bq' \neq 0$, these are equiv to [1'],[2']. In general, values of x that make $bq' - b'q = 0$ are not solns of [3] and [4] and neither is $y = 0$. Therefore, the sys:

$$(pb' - p'b)y + (b'q - bq') = 0 \qquad [3']$$
$$(b'q - bq')y + (p'q - pq') = 0 \qquad [4']$$

is equiv to [1'],[2']. Mult. [3'] by $(b'q - bq')$ and [4'] by $(pb' - p'b)$ and subtract:

$$(b'q - bq')^2 - (pb' - p'b)(p'q - pq') = 0 \qquad [5]$$

If $b'q - bq' \neq 0$, then [4'],[5] equiv to [3'],[4']. Expanding these, we get equivalents to the original system:

$$(b'(ax^2+2gx+c) - b(a'x^2+2g'x+c'))y$$
$$\quad - 4(b'(hx+f) - b(h'x+f'))(h'x+f')(ax^2+2gx+c) - (hx+f)(a'x^2+2g'x+c')) = 0 \qquad [6]$$
$$(b'(ax^2+2gx+c) - b(a'x^2+2g'x+c'))^2$$
$$\quad +2((h'x+f')(ax^2+2gx+c) - (hx+f)(a'x^2+2g'x+c')) = 0 \qquad [7]$$

[6] is a biquadratic giving 4 values of x
[7] is 1° in y and gives one value of y for each x

[6] is only reducible to a quadratic in the following cases and these cases are where the intersection of two conics are constructible with a ruler and compass.

1) If $b'/b = f'/f = c'/c$, [6] becomes

$$x^2(b'(ax+2g) - b(a'x+2g'))^2$$
$$\quad - 4(b'h-bh')((h'x+f')(ax+2g) - (hx+f)(a'x+2g') + (h'c-hc'))) = 0$$

where two roots are zero and two come from the quadratic.

2) If $a'/a = b'/b = h'/h$, the two highest terms disappear from [6]. Two of its roots become infinite and the other two come from the quadratic.

3) If $f,g,f',g' = 0$ then only even powers occur in [6] and the resultant becomes a quadratic in x^2.

4) In some cases, the resultant will be a reciprocal eqn.

Homogeneous systems of homogeneous eqns in x,y are often solved by letting $y = vx$.

Example

$x^2 + xy = 12 \qquad xy - 2y^2 = 1$

Let $y = vx$

$x^2(1 + v) = 12 \qquad x^2(v - 2v^2) = 1$
$\therefore\ x^2(1 + v) - 12x^2(v - 2v^2) = 0$
$\therefore\ x^2(24v^2 - 11v + 1) = 0$

Because $x=0$ not soln of original sys, original sys equiv to

$x^2(1+v) = 12 \qquad 24v^2 - 11v + 1 = 0$

From quadratic in v, $v = 1/3, 1/8$
From $v = 1/3$, $x = \pm 3$, from $1/8$, $x = \pm 4\sqrt{(2/3)}$
Solns: $(3,1)$ $(-3,-1)$ $(4\sqrt{(2/3)}, 1/\sqrt{6})$ $(-4\sqrt{(2/3)}, -1/\sqrt{6})$

Symmetrical systems, like sym. eqns, are unchanged by the exchange of any two vars. These are sym. sys:

Sys 1: $x + y = a \qquad x^2 + y^2 = b$
Sys 2: $x^2 + y = a \qquad y^3 + x = a$
Sys 3: $x - y + z = a \qquad yz + zx + xy = c \qquad x^2 + y^2 + z^2 = b$

The solns of such systems must also be symmetrical. If there are an even number of solns, say 4, and two are $x = a_1, a_2$ $y = b_1, b_2$, then the other two must be $x = b_1, b_2$ and $y = a_1, a_2$. If the number of solns is odd, the values of x,y must be equal, else there would be another sym. soln.

Example

System: [1]
$A(x^2 + y^2) + Bxy + C(x+y) + D = 1$
$A'(x^2 + y^2) + B'xy + C'(x+y) + D' = 1$

1st method
Let y = vx, eliminate x as above to get a resultant in v.

$$((D'A)+ (D'B)v + (D'A)v^2)^2 = (D'C)(1+v)^2((C'A) + (C'B)v + (C'A)v^2 \qquad [2]$$

where D'A ≡ D'A-DA', D'B ≡ D'B-DB', and so on. [2] is reciprocal and solvable by quadratics. The system comes down to (in this D'A notation):

$$((D'A) + (D'B)v + (D'A)v^2)x + (D'c)(1 + v) = 0 \qquad [3]$$

and is solvable from there.

2d method
sys [1] is equiv to

$$A(x+y)^2+ (B-2A)xy + C(x+y) + D = 0$$
$$A'(x+y)^2+ (B'-2A')xy + C'(x+y) + D' = 0$$

or

$$Au^2+ (B-2A)v + Cu + D = 0 \qquad \text{sys}[4]$$
$$A'u^2+ (B'-2A')v + C'u + D' = 0$$

Eliminate u^2 then v for equiv sys:

$$(A'B)v + (A'C)u + (A'D) = 0 \qquad \text{sys}[5]$$
$$(A'B)u^2+ ((C'B) - 2(C'A)u + ((D'B) -2(D'A)) = 0$$

where (A'B) etc. as above. Sys[5] has two solns, say u = a, a' v = b, b'. Then original system soln:

$$x = (a\pm\sqrt{(a^2- 4b)}/2, (a'\pm\sqrt{(a'^2- 4b')})/2$$
$$y = (a\mp\sqrt{(a^2- 4b)}/2, (a'\mp\sqrt{(a'^2- 4b')})/2$$

where both solns can be treated as:

$$2(x^2+y^2) - 3xy + 2(x+y) - 39 = 0$$
$$3(x^2+y^2) - 4xy + (x+y) - 40 = 0$$

General Theory of Integral Functions

We can view any ifn of form [*f*] as a product of its first coeff c_0 and its factors.

$$f = c_0(x - a_1)(x - a_2)\cdots(x - a_n)$$

where a_i are its n roots. Comparing this to the form and algebrating, we find:

$-c_1 = \sum$n roots taken one at a time
$c_2 = \sum$ product of n roots taken two at a time
…
c_n = product n roots

For if α,β roots of $ax^2+ bx + c = 0$ then $\alpha+\beta = -b/a$ and $\alpha\beta = c/a$ and if α,β,γ roots of $ax^3+ bx^2+ cx + d = 0$ then $\alpha+\beta+\gamma = -b/a$, $\beta\gamma+\gamma\alpha+\alpha\beta = c/a$ and $\alpha\beta\gamma = -d/a$. It follows that considering these coeff sums as s_i [1-r] of the 1 to r powers of roots α,β of the quadratic

$$x^2+ p_1x + p_2 = 0 \qquad [1]$$

that we can express s_i as integral fns of p_1 and p_2

$$s_1 = \alpha+\beta = -p_1 \qquad [2]$$
$$s_2 = \alpha^2+\beta^2 = (\alpha+\beta)^2- 2\alpha\beta = p_1^2- 2p_2 \qquad [3]$$

To get s_3, we know

$$\alpha^2+ p_1\alpha + p_2 = 0 \qquad \beta^2+ p_1\beta + p_2 = 0 \qquad [4]$$

Mult. 1st by α, 2d by β and add:

$$s_3 + p_1s_2 + p_2s_1 = 0 \qquad [5]$$
$$s_3 = -p_1(p_1^2 - 2p_2) + p_2p_1 = -p_1^3+ 3p_1p_2 \qquad [6]$$

Mult [4] by α^2, β^2 respectively:

$$s_4 = p_1s_3 + p_2s_2 = p_1^4- 4p_1^2p_2 + 2p_2^2$$

and so on. It follows that we can express any sym. ifn of the roots of quadratic [1] as an ifn of p_1 and p_2. And as α, β are any quantities whatever, then:

Thm. 3.20. ∀sym. fn of two vars can be expressed as a rational ifn of the two elementary sym. fns: $p_1= -(\alpha+\beta)$ and $p_2= \alpha\beta$.

Note: 1) If coeff of sym. fn ∈ **N** then coeff of equiv fn in p_1, p_2 ∈ **N**. 2) The sum of the suffixes (i.e. 1 for p_1) of the p's are equal to the degree of the sym. fn in α,β. As in [6]:

$$-p_1^3+ 3p_1p_2 \Rightarrow 1+1+1 = 3 = 1+2$$

and this sum is the **weight** of the sym. fn.

Examples

1) Calculate $\alpha^4 + \beta^4$ in terms of p_1, p_2

By note #2, $\alpha^4 + \beta^4 = Ap_1^4 + Bp_1^2p_2 + C\,p_2^2$ as homog. fn 4°. So A,B,C tbd.

Let $\beta=0 \Rightarrow p_1 = -\alpha \;\; p_2=0 \;\therefore\; \alpha^4 = A\alpha^4 \;\therefore\; A = 1$

$\therefore \alpha^4 + \beta^4 = (\alpha+\beta)^4 + B(\alpha+\beta)^2(\alpha\beta) + C\alpha^2\beta^2$

Term $\alpha^3\beta$ not on LHS $\therefore$ B = -4

Let $\alpha = -\beta = 1 \Rightarrow p_1 = 0 \;\; p_2 = -1 \;\therefore\; C = 2$

$\therefore \alpha^4 + \beta^4 = p_1^4 - 4p_1^2p_2 + 2p_2^2$

OR from $s_4 + p_1s_3 + p_2s_2 = 0$

$s_4 = -p_1(-p_1^3 + 3p_1p_2) - p_2(p1^2 - 2p_2)$

$\quad = p_14 - 4p_1^2p_2 + 2p_2^2$

2) Calculate $\alpha^5 + \beta^5 + \alpha^3\beta^2 + \alpha^2\beta_3$ in terms of p_1,p_2

As above $= Ap_1^5 + Bp_1^3p_2 + Cp_1p_2^2$

Let $\beta = 0 \Rightarrow A = -1$

For term $\alpha^4\beta$ we would have B = 5, let $\alpha = \beta = 1 \Rightarrow C = -6$

$\therefore = -p_1^5 + 5p_1^3p_2 - 6p_1p_2^2$

Because any alternating ifn of α,β vanishes if $\alpha = \beta$, it can be expressed as $(\alpha-\beta)$ times some sym. fn of α,β or as product of $\pm\sqrt{(p_1^2 - 4p_2)}$ and an ifn in p_1,p_2.

Example

Express $\alpha^5\beta - \alpha\beta^5$ in terms of p_1,p_2.

$= \alpha\beta(\alpha^4 - \beta^4)$

$= (\alpha - \beta)\alpha\beta(\alpha + \beta)(\alpha^2 + \beta^2)$

$= \pm\sqrt{(p_1^2 - 4p_2)}(p_1p_2(p_1^2 - 2p_2))$

Further, any sym. ifrac is expressible in terms of p_1,p_2:

$$\frac{\alpha^3 + 2\alpha^2\beta + 2\alpha\beta^2 + \beta^3}{\alpha^2\beta + \alpha\beta^2} = \frac{(\alpha+\beta)^3 - \alpha\beta(\alpha+\beta)}{\alpha\beta(\alpha+\beta)} = \frac{-p_1^3 + p_1p_2}{-p_2p_1} = \frac{p_1^2 - p_2}{p_2}$$

Newton extended this idea to sym. fns of any number of vars.

Thm. 3.21. Newton's Fundamental Theorem of Symmetric Functions

The sums of the integral powers of the roots of any ifn of form [*f*] with coeffs p_i can be expressed as ifns of p_i with coeff $\in$ **Z**.

Here the p_i are our c_i and using the same s_i as above, he derived the following two tables for calculating s_i in terms of p_i.

Table 1

$s_1 + p_1 = 0$

$s_2 + p_1s_1 + 2p_2 = 0$

$s_3 + p_1s_2 + p_2s_1 + 3p_2 = 0$

…

$s_{n-1} + p_1s_{n-2} + p_2s_{n-3} + \cdots + (n-1)p_{n-1} = 0$

Table 2

$s_n + p_1 s_{n-1} + \cdots + np_n = 0$
$s_{n+1} + p_1 s_n + \cdots + s_1 p_n = 0$
$s_{n+2} + p_1 s_{n+1} + \cdots + s_1 p_n = 0$
$\cdots$

Therefore, in expressions of s_r, the sum of the suffixes of p_i will be r. So to find all possible terms in s_r, we find all products of powers of p_i [1-n] where the sum of suffixes is r.

Example

Find the sum of the cubes of the roots of $x^3 - 2x^2 + 3x + 1 = 0$

$s_1 - 2 = 0 \quad s_2 - 2s_1 + 2\cdot3 = 0 \quad s_3 - 2s_2 + 3s_1 + 3\cdot1 = 0$
$\therefore\ s_1 = 2 \quad s_2 = -2 \quad s_3 = -13$

We can take this one step further:

Thm. 3.22. ∀sym. fn f in form [ƒ] of n vars x_i [1-n]. Using our sigma notation, let $\sum x_1$, $\sum x_1x_2$, …, $\sum x_1x_2\cdots x_n$ be the **elementary symmetric fns of x_i.** Then f is expressible as an ifn of these elementary fns and therefore, any rational sym. fn of these vars expressible as a rational fn of these n elementary sym. fns.

This idea will persist throughout the text. Let's do a couple of examples to see where this leads:

Examples

1) α,β,γ roots of $x^3 - p_1x^2 + p_2x - p_3 = 0$

Express $\beta^3\gamma + \beta\gamma^3 + \gamma^3\alpha + \gamma\alpha^3 + \alpha^3\beta + \alpha\beta^3$ in terms of p_i
$p_1 = \sum\alpha \quad p_2 = \sum\alpha\beta \quad p_3 = \sum\alpha\beta\gamma$
No term higher than 3° occurs in $\sum\alpha^3\beta$
$\therefore \sum\alpha^3\beta = Ap_1^2p_2 + Bp_1p_3 + Cp_2^2 \qquad [1]$
Let $\gamma = 0 \Rightarrow p_1 = \alpha+\beta \quad p_2 = \alpha\beta \quad p_3 = 0$
$[1] \Rightarrow \alpha^3\beta + \alpha\beta^3 = A(\alpha+\beta)^2\alpha\beta + C\alpha\beta \ \therefore\ A = 1 \ \ B = -2$
$\therefore \sum\alpha^3\beta = p_1^2p_2 + Bp_1p_3 - 2p_2^2$
Let $\alpha,\beta,\gamma = 1 \therefore p_1 = 3 \quad p_2 = 3 \quad p_3 = 1$
$\therefore 6 = 27 + 3B - 18 \ \therefore\ B = -1$
$\therefore \sum\alpha^3\beta = p_1^2p_2 - p_1p_3 - 2p_2^2$
$\therefore \sum\alpha^3\beta = (\sum\alpha)^2\sum\alpha\beta - \alpha\beta\gamma\sum\alpha - 2(\sum\alpha\beta)^2$

2) Eliminate x,y,z from system:

$x + y + z = 0 \qquad x^3 + y^3 + z^3 = a$
$x^5 + y^5 + z^5 = b \qquad x^7 + y^7 + z^7 = c$

Using our sums of powers: $s_3 = 3p_3 \quad s_5 = -5p_2p_3 \quad s_7 = 7p_2^2p_3$
So we need to eliminate p_2, p_3 from: $3p_3 = a \quad -5p_2p_3 = b \quad 7p_2p_3 = c$
This can be done at once. The result is $21b^2 - 25ac = 0$

Recall our analysis of quadratic roots in DME where, in form $ax^2+ bx + c = 0$, one or more of the coeffs vanished. We can tabulate these results for quadratics.

$\alpha,\beta \in$ **R**	$b^2-4ac > 0$	Roots opp sign	$c/a < 0$
$\alpha,\beta \in$ **C-R**	$b^2- 4ac < 0$	One root > 0	$c = 0$
$\alpha = \beta$	$b^2 - 4ac = 0$	Two roots = 0	$b,c = 0$
$\alpha = -\beta$	$b = 0$	One root ∞	$a = 0$
$\alpha,\beta > 0$	$c/a > 0 \;\; b/a < 0$	Two roots ∞	$a,b = 0$
$\alpha,\beta < 0$	$c/a,b/a > 0$		

We can generalize this for fns of form [*f*]. If the last r coeffs vanish then eqn has r infinite roots. Further, if $c_0= 0$, the sum of the roots equals zero. The condition that two roots, where roots are a_i [1-n], are equal is determined by expressing $\prod(a_1-a_2)^2$ (our sigma notation) in terms of p_i ([*f*]'s c_i) and equate this to zero.

Example

$x^3+ p_1x^2+ p_2x + p_3 = 0$ roots α,β,γ
$\prod(\beta-\gamma)^2 = (\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -4p_1^3p_3+p_1^2p_2^2+ 18p_1p_2p_3 - 4p_2^3 - 27p_3^2$
So the condition for equal roots is RHS = 0.
If all roots are real, RHS > 0. But if two are imaginary, RHS < 0.
As you can imagine, getting the RHS from the LHS was a grind.

Thm. 3.23 $\forall$quadratic ifn f(x) is completely determined when its roots are given and also the value of f for any x not a root.
Proof
Let roots of y be α,β $\therefore$ $y = A(x - \alpha)(x - \beta)$
Let $V = f(\lambda)$ $\therefore$ $V = (\lambda - \alpha)(\lambda - \beta)$ which determines A
$\therefore y = V\cdot ((x - \alpha)(x - \beta))/((\lambda - \alpha)(\lambda - \beta))$ ■
Cor. 1. $\forall$ifn uniquely determined by n+1 values V of $f(\lambda_i)$ [1-(n+1)]
Proof
$f = ax^2+ bx + c$, λ_i [1-3]
$a\lambda_1^2+ b\lambda_1+ c = V_1$ $a\lambda_2^2+b\lambda + c = V_2$ $a\lambda_3^2+ b\lambda_3 + c = V_3$
This linear system uniquely determines a, b, c.
$$\therefore y = V_1\frac{(x - \lambda_2)(x - \lambda_3)}{(\lambda_1- \lambda_2)(\lambda_1- \lambda_3)} + V_2\frac{(x - \lambda_1)(x - \lambda_3)}{(\lambda_2- \lambda_1)(\lambda_2- \lambda_3)} + V_3\frac{(x - \lambda_1)(x - \lambda_2)}{(\lambda_3- \lambda_1)(\lambda_3- \lambda_2)}$$
which is all general in reasoning. ■

And so, for n+1 values, we have **Lagrange's Interpolation Formula**

$$y = \sum V_1((x - \lambda_2)(x - \lambda_3)\cdots(x - \lambda_{n+1}))/((\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_{n+1}))$$

I should point out that this older sigma notation can be distinguished from modern summation notation by its not having indices. It is sigma of $\sum V_1\cdot$etc. where the modern would be $\sum V_i\cdot$etc. [1-(n+1)]. As soon as this dawns on you, the two are no longer confusing.

Examples

1) Required eqn of min°, coeff $\in$ **Q** with root $\sqrt{2}+\sqrt{3}$.
We know other roots must be $\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$, $-\sqrt{2}-\sqrt{3}$
$\prod(x - \sqrt{2} - \sqrt{3}) = x^4 - 10x^2 + 1 = 0$

2) Required quadratic f(x): f or 2,3,4 = 4,4,6.

$$\therefore\ 4\frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} + 4\frac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} + 6\frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)}$$
$$= x^2 - 3x + 6$$

The condition that two quadratics have a common root is the same as their having a common factor which is to say that their GCM is not equal to a constant. So if two ifns have r roots in common, their GCM is r° and its coeffs are rational fns of the coeffs of the two original ifns. And if r is odd, at least one root is real.

Consider these forms of $y = ax^2 + bx + c$ roots α,β $a>0$ [1]

1. $y = a((x - l)^2 - m)$ $\alpha = l + \sqrt{m}$ $\beta = l - \sqrt{m}$
2. $y = a(x - l)^2$ $\alpha = \beta = l$
3. $y = a((x - l)^2 + m)$ $\alpha = l + i\sqrt{m}$ $\beta = l - i\sqrt{m}$

Here, $l,m \in \mathbf{R}$ and $m>0$. In all cases, as $x \to \infty$, $(x - l)^2 \to \infty$ $\therefore$ y infinite when x infinite and y has the sign of a. Excluding the factor a, as $x \to 0$, $y \to 0$ when $x>l$; y is min when $x=l$, and y increases as $x<l$. Therefore, y has max or min when $x = l$ as a is neg or pos. In #1, roots real and unequal then y same or different sign as a as x does or does not lie between the roots. In the other two, y always has the same sign as a. If from [1] we derive

$$ax^2 + bx + (c - y) = 0 \qquad [2]$$

we have equal roots when $b^2 - 4a(c - y) = 0$ or

$$y = -(b^2 - 4ac)/4a \ \therefore x = -b/2a \qquad [3]$$

If $y<0$ in [3], y is min below X-axis. If $y=0$, this min is on the X-axis. And if $y>0$, min is above the axis. If $a<0$, these mins are maxs. If we view $y = 2x^2 - 12x + 13$ in this way

$$y = 2(x^2 - 6x + 9) - 5$$
$$= 2((x - 3)^2 - 5/2)$$
$$= 2(x - (3 - \sqrt{(5/2)})(x - (3 + \sqrt{(5/2)})$$

Therefore it is type #1 and y is min when $x = 3$ $\therefore$ min at $y = -5$. All of this gives us a kind of pre-Calculus algebraic approach to problems of maxima and minima. So more generally, we can express the condition for any $y = f(x)$ that the roots of $f(x) - y = 0$ are equal being that we have a max or min, as by increasing or decreasing y from that point the roots are lost.

Examples

1) $y = x^3 - 9x^2 + 24x + 3 \therefore x^3 - 9x^2 + 24x + (3 - y) = 0$ roots α_i [1-3]
Let D denote $\prod(\alpha_1 - \alpha_2)$ then roots real, two equal, two imaginary as D >=< 0.
Using our $D = -4p_1^3 p_3 + \cdots$ above, $p_1=-9$, $p_2= 24$, $p_3= (3-y) \therefore D = -27(y-19)(y-23)$
$\therefore$ y = 19,23 are max or min. By testing, 19 min and 23 max.
At max or min, two roots are equal: α,α,γ. So we can calculate x as α.
$2\alpha+\gamma = 9 \quad \alpha^2 + 2\alpha\gamma = 24 \quad \alpha^2 - 6\alpha + 8 = 0 \therefore \alpha = 2,4 \therefore$ (4,19) min and (2,23) max

2) Analyze $(x^2 - 7x + 6)/(x^2 - 8x + 15)$
First we derive a quadratic for x in terms of y:
$(1 - y)x^2 - (7 - 8y)x + (6 - 15y) = 0$ [1]
$\therefore D = (7 - 8y)^2 - 4(1 - y)(6 - 15y) = 4(y - (7/2 - \sqrt{6}))\cdot(y - (7/2 + \sqrt{6}))$
$\therefore$ max/min of $y = 7/2 \pm \sqrt{6}$
$x = \frac{1}{2}(7 - 8y)/(1-y) \therefore x = 9 \pm 2\sqrt{6}$
(Here, x came from your high-school quadratic equation)
Further y is discontinuous at x = 3,5, and when $x = \infty$, y = 1
and y also equals 1 at x = 9.

You should graph this last eqn. We know from our elementary Calculus in DME that these max/min are the roots of f '(x). But here, this method is probably quicker than taking the derivative.

4. Series

Progressions

We begin by extending our understanding of arithmetic and geometric progressions and their related simple series and sums.

Def. The sum of n terms formed by some law f(n) $n \in \mathbf{N}$ is a **series** and takes form:

$$f(1) + f(2) + \cdots + f(r) + \cdots + f(n) \qquad [1]$$

The rth term here is the **general term**.

Example

$f(n) = n^2 + 2n$
$(1^2 + 2\cdot1) + (2^2 + 2\cdot2) + (3^2 + 2\cdot3) + \cdots + (n^2 + 2n)$
$= 3 + 8 + 15 + \cdots + (n^2 + 2n)$

If we consider [1] as some φ(n) then φ has the property that the number of its terms depends upon the value of its var. We've had arithmetic series where a is the first term and b is the common difference:

$$a \quad a + b \quad a + 2b \quad \cdots \quad a + nb$$

and here the sum was $\sum = n/2(2a - (n-1)b)$ or if l is the last term, $\sum = n\cdot(a+l)/2$.

Examples

1) Sum $5 + 3 + 1 + -1 + \cdots$ to 100 terms.
$100/2(2\cdot5 + (100 - 2)\cdot-2) = 50(10 - 198) = -9400$

2) Sum first n odd natural numbers
$\sum = 1 + 3 + 5 + \cdots + (2n - 1) = n\cdot(1 + (2n - 1))/2 = n^2$

3) Sum $1 - 2 + 3 - 4 + 5 + \cdots + (2m - 1) + 2m$ to n terms
n even) $= 2m \therefore 1 - 2 + 3 - 4 + \cdots + (2m - 1) + 2m$
$= 1 + 3 + 5 + \cdots + 2m - 1$
$- 2 - 4 - 6 - \cdots - 2m$ where each line has m terms
Line 1 $= m^2$ by #2 Line 2 $= -m(2 + 2m)/2 = -m(m+1)$
$\therefore \sum = m^2 - m(2 + 2m)/2 = -m = -n/2$
n odd) $= 2m - 1 \therefore \sum = 1 - 2 + 3 - \cdots + 2m - 1$
or last result $+ 2m = m = (n+1)/2$

Let's develop a general method as the above requires each term to be of same degree.

Let the nth term of a series be:

$$p_0 n^r + p_1 n^{r-1} + p_2 n^{r-2} + \cdots + p_r$$

where p_i is independent of n. Let's denote the sums of the first to rth powers of the first $n \in \mathbf{N}$ as n^{s1} to n^{sr}:

$$n^{s1} = 1 + 2 + 3 + 4 + \cdots + n$$
$$n^{s2} = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$
$$\cdots$$

Let $\sum$ denote the sum of n terms of a series where [1] is the nth term:

$$\sum = p_0 n^{sr} + p_1 n^{sr-1} + p_2 n^{sr-2} + \cdots + np_n$$

where, for example, 1st term = $p_0 1^r + p_0 2^r + \cdots + p_0 n^r$ etc. So we need to calculate n^{si}:

$\mathbf{n^{s1}}$ In $(x + 1)^2 - x^2 = 2x + 1$ we let x = n, n-1, ...

$$(n + 1)^2 - n^2 = 2n + 1$$
$$(n^2) - (n - 1)^2 = 2(n-1) + 1$$
$$\cdots$$
$$3^2 - 2^2 = 2 \cdot 2 + 1$$
$$2^2 - 1^2 = 2 \cdot 1 + 1$$

adding:

$$(n + 1)^2 - 1 = 2n^{s1} + n$$
$$2n^{s1} = (n + 1)^2 - (n + 1) = (n+1)n$$
$$n^{s1} = ((n+1)n)/2$$

∴ Sum 1st powers of first $n \in \mathbf{N}$ is a 2° ifn f(n)

$\mathbf{n^{s2}}$ Sym. $(x + 1)^3 - x^3 = 3x^3 + 3x + 1$ and again x = n, n-1, ... and we add:

$$(n + 1)^3 - 1 = 3n^{s2} + 3n^{s1} + n$$
$$3n^{s2} = (n + 1)^3 - 3/2\, n(n + 1) - (n + 1) = (n+1)/2 \cdot (2n^2+n)$$
$$n^{s2} = 1/6 \cdot n(n + 1)(2n + 1)$$

∴ Sum squares of first $n \in \mathbf{N}$ is a 3° ifn f(n)

$\mathbf{n^{s3}}$ Sym. using $(x + 1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

$$n^{s3} = \left((n(n+1))/2 \right)^2$$

∴ Sum cubes of first $n \in \mathbf{N}$ is 4° ifn f(n) and the square of the sum of their first powers.

∴ In general, n^{sr} is an ifn f(n) of (r+1)° of form:

$$n^{sr} = q_0 n^{r+1} + q_1 n^r + q_2 n^{r-1} + \cdots + q_{r+1}$$

Further n^{sr} divby n(n+1)

$$n^{sr} = n(n+1)(n^{r-1}/(r+1) + p_1 n^{r-2} + p_2 n^{r-3} + \cdots + p_{r-1})$$

We can now sum any series where the nth term is an ifn f(n). Because it is an ifn f(n) of (r+1)° it must have form:

$$An^{r-1} + Bn^{r} + \cdots + K$$

and by giving values to n we determine A, B, ..., K. If S_i [1-(r+2)] are the sums of the 1, 2, ..., (r+2) terms of the series, then by Lagrange above, the sum is

$$\sum S_i \cdot \frac{(x-1)(x-2)\cdots(x-i+1)(x-i-1)\cdots(x-r-2)}{(i-1)(i-2)\cdots 1\cdot -1\cdots(i-r-2)} \quad [1-(r+2)]$$

Examples

1) Sum a + (a+b) + (a+2b) + ⋯ + (a + (n-1)b)

a - b + nb
a - b + (n-1)b
…
a - b + 2b
a - b + b
$n(a - b) + n^{s1}b = (a-b)n + b\cdot(n(n+1)/2 = n/2\cdot(2a + (n-1)b)$

But we knew that.

2) $\sum = 1^2 + 3^2 + 5^2 + \cdots$ n terms. The nth term = $(2n - 1)^2$ =

$4n^2 - 4n + 1$
$4(n-1)^2 - 4(n-1) + 1$
…
$4\cdot 2^2 - 4\cdot 2 + 1$
$4\cdot 1^2 - 4\cdot 1 + 1$
$4n^{s2} - 4n^{s1} + n$

∴ ((2n - 1)n(2n + 1))/3

3) $\sum = 2\cdot3\cdot4 + 3\cdot4\cdot5 + 4\cdot5\cdot6 + \cdots + (n+1)(n+2)(n+3)$
∴ nth term = $n^3 + 6n^2 + 11n + 6$
∴ $\sum = n^{s3} + 6n^{s2} + 11n^{s1} + 6n$
∴ = $\frac{1}{4}(n^4 + 10n^3 + 35n^2 + 50n)$

We can find the general soln for a geometric series as well. The nth term is

$$(p_0 n^s + p_1 n^{s-1} + p_2 n^{s-2} + \cdots + p_s)r^n$$

where p_i is independent of n and s∈**N**. Our DME G.P. is the simplest case of this where the nth term is

$$p_s r^n \quad \text{or} \quad p_s r\cdot r^{n-1} \quad \text{or} \quad ar^{n-1} : a = p_s r \text{ is constant}$$

Here the ratio of each term to its antecedent is r ≡ constant.

$$\sum = a + ar + ar^2 + \cdots + ar^{n-1} \qquad [1]$$

Mult. both sides by (1-r)

$$(1-r)\sum = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$- ar - ar^2 - \cdots - ar^{n-1} - ar^n$$
$$= a - ar^n$$
$$\therefore \sum = (a - ar^r)/(1 - r) \text{ or if l is last term and } ar^n = rl, \sum = (a - rl)/(1 - r)$$

Examples

1) $\sum = 3/2 + 3/4 + 3/8 + \cdots$ for 10 terms
$a = 3/2 \quad r = 1/2$
$\sum = 3/2 \cdot (1 - (1/2)^{10})/(1 - 1/2) = 3(1 - 1/2^{10})$

2) $\sum = 1 - 2 + 4 - 8 + \cdots$ for n terms
$a = 1 \quad r = -2$
$\sum = 1 \cdot (1 - (-2)^n)/(1 - (-2)) = (1 - (-1)^n 2^n)/3$
$\therefore$ n even $\sum = 1/3(1 - 2^n)$ n odd $\sum = 1/3(1 + 2^n)$

3) $\sum (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots$ for n terms
$= \frac{x^2 - y^2}{x - y} + \frac{x^3 - y^3}{x - y} + \frac{x^4 - y^4}{x - y} + \cdots + \frac{x^{n+1} - y^{n+1}}{x - y}$
$= 1/(x-y) \cdot (x^2 + x^3 + \cdots + x^{n+1}) - 1/(x-y) \cdot (y^2 + y^3 + \cdots + y^{n+1})$
$= x^2/(x-y) \cdot (1 + x + \cdots + x^{n-1}) - y^2/(x-y) \cdot (1 + y + \cdots + y^{n-1})$
$1 + x + \cdots + x^{n-1} = (1 - x^n)/(1 - x)$
$1 + y + \cdots + y^{n-1} = (1 - y^n)/(1 - y)$
$\sum = \frac{x^2(1 - x^n)}{(x-y)(1-x)} - \frac{y^2(1 - y^n)}{(x-y)(1-y)}$

Now instead of a constant, let a be an ifn, f(n) 1°, general term:

$$(a + bn)r^n\text{: a,b constants} \qquad [1]$$

Note that this form would arise if each term of an A.P. series were mult. by its corresponding term in a G.P. series. So this is an **arithmeticogeometric series**, if anyone asks.

$$\sum = (a + b\cdot 1)r^1 + (a + b\cdot 2)r^2 + \cdots + (a + b\cdot n)r^n$$

Mult. both sides by (1 - r)

$$(1 - r)\sum = (a + b\cdot 1)r^1 + (a + b\cdot 2)r^2 + \cdots + (a + b\cdot n)r^n$$
$$- (a + b\cdot 2)r^2 - \cdots - (a + b\cdot(n-1))r^n - (a + b\cdot n)r^{n+1}$$
$$= a + br + br^2 + \cdots + br^n - (a + bn)r^{n+2}$$

We mult. both sides by (1 - r) again. Verify that this yields:

$$(1 - r)^2\sum = (a+b) - (a+b)r + br^2 - (a + (n+1)b)r^{n+1} + (a + bn)r^{n+2}$$
$$\therefore \sum = \text{RHS}/(1 - r)^2$$

So what's the deal with multiplying by (1 - r)?

Let $f_s(n)$ be an ifn in n of s°. Then $f_s(n) - f_s(n-1)$ is (s - 1)° as you can work out by using the series terms above. Then $f_s(n-1) - f_s(n-2)$ is (s - 2)° and so on. So we have the series:

$$\sum = f_s(1)r + f_s(2)r^2 + \cdots + f_s(n)r^n \qquad [1]$$

$$\begin{aligned}(1 - r)\sum &= f_s(1)r + f_s(2)r^2 + \cdots + f_s(n)r^n \\ &\quad - f_s(1)r^2 - \cdots - f_s(n-1)r^n - f_s(n)r^{n+1} \\ &= f_s(1)r + f_{s-1}(2)r^2 + \cdots + f_{s-1}(n-1)r^n + f_s(n)r^{n+1}\end{aligned}$$

If we exclude first and last terms, we have a simple series of lower degree. If we mult. [1] by $(1 - r)^{s+1}$ this simple series would vanish, leaving only a fixed number of terms yielding a formula for ∑.

Example

$\sum = 1^2r + 2^2r^2 + 3^2r^3 + \cdots + n^2r^n$

r^n is multiplied by a term of 2° ∴ mult. both sides by $(1 - r)^3$ to get

$$\sum = \frac{r + r^2 - (n+1)^2r^{n+1} + (2n^2 + 2n + 1)r^{n+2} - n^2r^{n+3}}{(1 - r)^3}$$

If the last term had been $-n^2r^n$, you would mult. both sides by $(1 + r)^3$.

Recall from DME that if x ∈ (0,1) then $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$.

In G.P., $\sum = a + ar + ar^2 + \cdots + ar^{n-1}$

r = 1 $\sum = a + a + a + \cdots + a = na$ ∴ n→∞ ∑→∞

r > 1 $\sum = \frac{a(r^n - 1)}{r - 1} = \frac{ar^n}{r - 1} - \frac{a}{r - 1}$ ∴ Again, n→∞ ∑→∞

r ∈ (0,1) $\sum = ar^n/(r-1) - a/(r-1)$ ∴ n→∞ LHT→0 RHT→a/(1-r)

r = -1 $\sum = a - a + a - \cdots$ ∴ n→∞ ∑ **oscillates** between a and 0.

Examples

1) $\sum = 1/2 + 1/2^2 + 1/2^3 + \cdots$

$$\sum = 1/2 \cdot (1 - 1/2^n)/(1 - 1/2) = 1 - 1/2^n \quad \therefore n\to\infty \ \sum\to 1$$

2) Evaluate 0.34343434...

$$\sum = 34/100 + 34/100^2 + 34/100^4 + \cdots = 34/100 \cdot 1/(1 - 1/100) = 34/99$$

For A.P., given $\sum = n/2 \cdot (2a + (n - 1)b)$, if we have three of $\sum$, a, b, n, we can solve for the fourth. When n is unknown, the eqn is a quadratic.

Examples

1) $\sum = 36 \;\; a = 15 \;\; b = -3$

$\therefore 36 = n/2 \cdot (30 - (n - 1)3)$

$\therefore n^2 - 11n + 24 = 0$ soln: 3,8

Verify that the sum of the first 8 terms equals the sum of the first 3 terms in order to understand the soln.

2) $\sum = 14 \;\; a = 3 \;\; b = 2$

$\therefore n^2 + 2n = 14 \;\; \therefore n = -1 \pm \sqrt{13} \cong 2.87, -4.87$

The negative root is outside the problem space. The fractional value of 2.87 shows that the series does not sum to 14. But it will sum to its values nearest 14 by taking the nearest values in **N** to 2.87: 2,3.

An A.P. is determined by its first term and its common difference which makes an A.P. a **twofold manifoldness**: determined by two independent data.
So we can write 3, 4, 5, … terms of an A.P. in a general way:

$\alpha-\beta \quad \alpha \quad \alpha+\beta$

$\alpha - 3\beta \quad \alpha-\beta \quad \alpha+\beta \quad \alpha+3\beta$

$\alpha-2\beta \quad \alpha-\beta \quad \alpha \quad \alpha+\beta \quad \alpha+2\beta$

…

For an odd number of terms, common difference is β, for even terms 2β.

Example

If a,b,c in A.P. then $a^2(b+c) + b^2(c+a) + c^2(a+b) = 2/a \cdot (a + b + c)^2$ [1]
Let $a = \alpha-\beta \;\; b = \alpha \;\; c = \alpha+\beta$ then [1] becomes

$$(\alpha-\beta)^2(2\alpha+\beta) + \alpha^2 2\alpha + (\alpha+\beta)^2(2\alpha-\beta) = 2/9 \cdot (3\alpha^3) = 6\alpha^3$$

$$\text{LHS} = 2\alpha((\alpha-\beta)^2 + (\alpha+\beta)^2) + \beta((\alpha-\beta)^2 - (\alpha+\beta)^2) + 2\alpha^3$$

$$= 2\alpha(2\alpha^2 + 2\beta^2) + \beta(-4\alpha\beta) + 2\alpha^3$$

$$= 6\alpha^3$$

If a,b,c in A.P, then $b - a = c - b \;\therefore b = (c+a)/2$. Here b is the **arithmetic mean** of a,c.
$\forall a,c \; \forall A_i$ [1-n]: a, A_1, A_2, …, A_n, c in A.P. then A_i are n arithmetic means between a,c. A_i are calculated by $A_1 = a+b$, $A_2 = a+2b$, …, $c = a + (n+1)b \;\therefore\; b = (c-a)/(n+1)$
$\therefore A_1 = a + (c-a)/(n-1) \quad A_2 = a + 2(c-a)/(n-1)$ and so on.
The arithmetic mean of n quantities a_i [1-n] is $(\sum a_i)/n$.

We have the sum of n terms in G.P.: $\sum = a(r^n - 1)/(r - 1)$. So if three if $\sum$,a,r,n are given, the fourth is determined. If r unknown, we have an ifn of n° and soln will be approximate for degree > 4. If n unknown, we have an exponential eqn $r^n = s$, where r,s known. This is solved by logarithms. G.P. is also a twofold manifoldness.

Example

If a,b,c,d in G.P. then

$$4(a^2+ b^2+ c^2+ d^2) - (a + b + c + d)^2 = (a-b)^2+ (c-d)^2+ 2(a-d)^2$$

Let $b = ra \quad c = r^2a \quad d = r^4a \;\therefore$

$$4a^2(1 + r^2+ r^4+ r^6) - a^2(1 + r + r^2+ r^3) = a^2(1-r)^2 + a^2r^4(1-r)^2 + 2a^2(1-r^3)^2$$
$$4(1 + r^2+ r^4+ r^6) - (1 + 2r + 3r^2+ 4r^3+ 3r^4+ 2r^5+ r^6)$$
$$= 1 - 2r + r^3 + r^4 - 2r^5 + r^6+ 2 - 4r^3 + 2r^6$$

When a,b,c in G.P., b is the **geometric mean** of a,c. Here a:b::b:c $\therefore b^2 = ac \therefore b = \pm\sqrt{ac}$. Again, between a and c we can insert n geometric means G_i [1-n] where r is the common ratio $\therefore$ $G_i = ar^i$ and $c = ar^{n+1}$ $\therefore$ $r = (c/a)^{1/(n+1)}$. So the geometric mean of n positive quantities is the positive value of the nth root of their product.

If a,b,c in A.P., then 1/a,1/b,1/c are in **Harmonic Progression** or **H.P.**

Thm. 4.1. If a,b,c in H.P. then a/c = (a-b)/(b-c)
Proof
1/a,1/b,1/c in A.P. $\therefore$ 1/b - 1/a = 1/c - 1/b $\therefore$ (a-b)/ab = (b-c)/bc
$\therefore$ (a-b)/(b-c) = ab/bc = a/c ■

H.P. is also a twofold manifoldness and the general form can be taken from that of A.P. above, as in

$$1/(\alpha-\beta) \qquad 1/\alpha \qquad 1/(\alpha+\beta)$$

and so on. If a,b,c in H.P., then b is the **harmonic mean** of a,c. From

$$1/c - 1/b = 1/b - 1/a \Rightarrow 2/b = 1/a + 1/c \;\therefore b = 2ac/(a+c)$$

And again, n harmonic means H_i [1-n] can be inserted between a and c where the common difference

$$d = (1/c - 1/a)/(n+1) = (a-c)/((n+1)ac)$$

$$\therefore \qquad 1/H_i = 1/a + (a-c)/(i - (n+1)ac)$$
$$\therefore \qquad H_1= ((n+1)ac)/(a + nc) \quad H_2= ((n+1)ac)/(2a+(n-1)c) \text{ and so on.}$$

The geometric mean between two positive reals a,c is the geometric mean between the arithmetic and harmonic means of a,c. And the arithmetic, geometric, and harmonic means are in descending order of magnitude:

$$A = (a+c)/2 \quad G = +\sqrt{ac} \quad H = 2ac/(a+c)$$
$$\therefore\ AH = (a+c)/2 \cdot 2ac/(a+c) = ac = G$$

And you can prove this descending order by comparing A-G and G-H. You can also prove that the sum of an harmonic series of n terms cannot be expressed by any rational algebraic function of n. Therefore an harmonic series cannot be summed.

Euclid's Means

Given AB<AC find the three means.

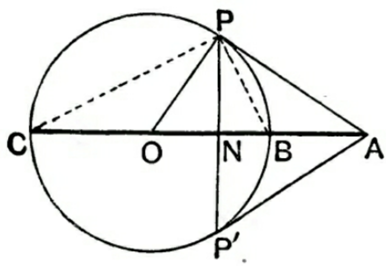
Method

ACB collinear. BC ×/2 @ O

⊙O,OB AP,AP' tan to ⊙O @ P,P'

PP' × AC @ N

Then 1) AO is arithmetic mean, 2) AP is geometric mean, and 3) AN is harmonic mean.



Proof

1) AC - AO = OC = BO = AO - AB

∴ 2AO = AB + AC ∴ AO = (AB + AC)/2 ≡ AO arithmetic mean

2) AP^2 = AB•AC (Euclid 3.36) ∴ AB : AP :: AP : AC ≡ AP geometric mean

3) Note: In pure geometry, the harmonic mean is defined by the proportion:

1st line : 2d line :: excess of 1st over harmonic mean : excess of harmonic mean over 2d

∠APB = ∠PCB ∴ ΔAPB ~ ΔACP

∴ AB : BP :: AP : PC

∴ $AB^2 : BP^2 :: AP^2 : AP^2$

∴ AB^2 : AB•AC :: BN•BC : CN•CB (Euclid 3.36, 6.8)

∴ AB : AC :: BN : NC

∴ AC : AB :: NC : BN :: AC-AN : AN-AB ≡ AN harmonic mean ■

If you are curious, the pure geometric definition of harmonic mean is equivalent to our algebraic one.

Inequalities

Inequalities are important in understanding limits and infinite sums and products. There are many expressions we cannot evaluate in these domains of thought. But if we can trap them between two expressions we can evaluate, and the outside expressions converge to the same value, our mystery expression must do the same. We are dealing only with real quantities to avoid the issue of complex orderings. On the real number line (-∞,+∞), a < b means a is to the left of b. This is the geometric sense of our ordering of the reals. In the following theorems we have LHS > RHS and all can be symmetrically adjusted to reveal the truth of RHS < LHS. But you knew that.

Thm. 4.2. $P>Q \wedge Q>R \Rightarrow P>S$

Proof

$P - Q + Q - R = R - S$

$P - Q, Q - R > 0 \therefore P - R > 0 \therefore P > R$ ■

Thm. 4.3. $P > R \Rightarrow P \pm Q > Q \pm R$

Cor. 1. $P+Q > R+S \Rightarrow (P + Q - R > S) \wedge (-R - S > -P - Q) \wedge (-P - Q < -R - S)$

Cor. 2. $\forall$inequality reduces to form $P > 0 \vee P < 0$

Thm. 4.4. $P_1>Q_1, P_2>Q_2, \ldots, P_n>Q_n \Rightarrow \sum P_i > \sum Q_i$

Note: You can prove this with Thm. 4.2. but $P_1>Q_1 \wedge P_2>Q_2 \; !\Rightarrow P_1 - P_2 > Q_1 > Q_2$.

Thm. 4.5. $P > Q \Rightarrow (PR > QR) \wedge (P/R > Q/R)$ if $R > 0$. If $R < 0 \Rightarrow (PR < QR) \wedge (P/R < Q/R)$

Cor. 1. $(P > QR) \wedge (R > S) \Rightarrow (P > QS)$ if $Q > 0$

Cor. 2. Any fractional inequality can be integralized.

Thm. 4.6. $P_1>Q_1, P_2>Q_2, \ldots, P_n>Q_n \wedge \forall P_i,Q_i > 0 \Rightarrow \prod P_i > \prod Q_i$

Cor. 1. $\forall n\in\mathbf{N}, (P > Q) \Rightarrow (P^n > Q^n)$

Cor. 2. $\forall n\in\mathbf{N}, (P,Q > 0) \wedge (P > Q) \Rightarrow (P^{1/n} > Q^{1/n})$ if we take the real positive nth root.

Cor. 3. $\forall r\in\mathbf{R}, (P,Q > 0) \wedge (P > Q) \Rightarrow (P^{-r} > Q^{-r})$

Cor. 4. Any inequality can be rationalized if process is governed by the above restrictions on signs.

Examples

1) When is $F = (3x - 4)/(x - 2) >< 1$?

$(\times (x-2)^2)$ $\quad (3x - 4)(x - 2) >< (x - 2)^2$

$((3x - 4) - (x - 2))(x - 2) >< 0$

$2(x - 1)(x - 2) >< 0$

$F > 1$ if $x < 1$ or $x > 2$

$F < 1$ if $1 < x < 2$

2) When is $x^3 + 25x >< 8x^2 + 26$

$x^3 - 8x^2 + 25x - 26 >< 0$

$(x - 2)(x^2 - 6x + 13) >< 0$

$(x - 2)((x - 3)^2 + 4) >< 0$

$\forall x$, RHT $> 0 \therefore x^3 + 25x >< 8x^2 + 26$ as $x >< 2$

3) Show $\forall x,y,z \in \mathbf{R}$, not all equal, then $\sum x^3 >< \sum 3xyz$ as $\sum x >< 0$

$\sum x^3 - \sum 3xyz = \sum x(\sum x^2 - \sum xy)$

$= \tfrac{1}{2}\sum x \sum (x - y^2)$

Thm is true as $\sum (x - y^2) > 0$ ■ I included this last one for its art of sigma-notation. The following theorems are all derived from the earlier ones on inequality.

Thm. 4.7. If b_i [1-n] > 0 the fraction $\sum a_i/b_i$ [1-n] is not less than the least nor greater than the greatest of the fractions a_i/b_i.

Proof

Let f, f' be least and greatest of these fractions. Then no $a_i/b_i < f$

$\forall i, b_i > 0 \therefore a_i \geq fb_i \therefore \sum a_i \geq \sum fb_i \therefore \sum a_i/\sum b_i \geq f$. Sym. for f'. ■

Note in these, that because i is in both indices, they are the same. You would use i,j to show they differ. Proofs of the following are similar to that last one.

Thm. 4.8. a_i,b_i as before, l_i [1-n] > 0 then $\sum l_i a_i/\sum l_i b_i$ is not less than the least or greater than the greatest of the a_i/b_i.

Thm. 4.9. $a_i,b_i,l_i > 0$ the $(\sum l_i a_i^m/\sum l_i b_i^m)^{1/m}$ and $(\prod a_i/\prod b_i)^{1/n}$ are not less than least etc.

Example

Show $\frac{1}{2} < ((1\cdot3\cdot5\cdots(2n-1))/(2\cdot4\cdots2n))^{1/n} < 1$

By Thm. 4.9. $\frac{1}{2}$ < middle term < $(2n-1)/2n$

and $(2n-1)/2n = 1 - 1/2n < 1$

Thm. 4.10. $x,p,q > 0$ $p,q \in \mathbf{N} \Rightarrow (x^p - 1)/p >< (x^q - 1)/q$ as $p >< q$

Thm. 4.11. $x>0, x\neq1 \Rightarrow mx^{m-1}(x - 1) > x^m - 1 > m(x - 1)$

unless $m \in (0,1) \Rightarrow mx^{m-1}(x - 1) < x^m - 1 < m(x - 1)$

Cor. 1. If $x,y > 0$ $x\neq y$ we can replace x with x/y, multiply by y^m and get

$mx^{m-1}(x - y) > x^m - y^m > my^{m-1}(x - y)$

in the first case and flip > to < in the second.

The proofs of 4.10-11 are longish and these theorems are important. But you need them more as tools than you need their explanations. However, if you are interested in inequalities, these longer proofs are the best source of technique for handling inequalities. You get to see people like De Morgan, Abel, Gauss, Euler, and Cauchy at work in many of the proofs using inequalities here and with limits and series. The masters are the best teachers if your road is their road.

Having mentioned Abel, a brief word on the progressive white-washing of mathematicians's morals. Long story short, Gauss is morally responsible for Abel's death. And the French Academy is responsible for Galois's death. And the textbooks all gloss this over. The gloss gets shinier over time. But the French Academy learned, to some extent, its lesson. When Hermites came before it, they understood him no better than they had Galois. But someone there stood up for him, I forget who, in part by saying he wouldn't be responsible for another Galois. Being brilliant does not expiate all sin. And Gauss never learned his lesson. He was a very poor human being. As mathematics is an anthropological -- that is to say, human -- activity, being a good mathematician requires expressing a good humanity. Or you're just a prick, like Gauss.

Thm. 4.12. Given n positive quantities, their arithmetic mean is greater than or equal to their geometric mean.

Proof

Consider the geom. mean of n quantities $(abcd\cdots k)^{1/n}$. If a-k not all equal replace the least and greatest, say a,k, with (a+k)/2. Then as $((a+k)/2)^2 > ak$, the geom. mean is increased but the arith. mean is unchanged. Until all are equal, wash, rinse, repeat. So the geom. mean constantly approaches and at last equals the arith. mean. ■

Cor. 1. If a-k are n pos. quantities and p-t are n quantities $\in$ **Q** $\Rightarrow$

$$\frac{pa + qb + \cdots + tk}{p + q + \cdots + t} \geq (a^p b^q \cdots k^t)^{1/(p + q + \cdots + t)}$$

Example

Show $1\cdot3\cdots(2n - 1) < n^n$

$(1 + 3 + \cdots + 2n - 1) > (1\cdot3\cdots(2n-1))^{1/n}$

$\therefore n^2/n > RHS \;\therefore\; n^n > 1\cdot3\cdots(2n - 1)$

Thm. 4.13. a-k n pos. quantities, p-t n pos. quantities $\Rightarrow$

$$(pa^m + qb^m + \cdots + tk^m)/(p + q + \cdots + t) \;\geq\leq\; [\,(pa + qb + \cdots + tk)/(p + q + \cdots + t)\,]^m$$

as $m \in$ or $\notin (0,1)$

Cor. 1. If p-t all equal $\Rightarrow$

$$(a^m + b^m + \cdots + k^m)/n \;\geq\leq\; [\,(a + b + \cdots + k)/n\,]^m$$

as $m \in$ or $\notin (0,1)$

OR, this is the relation of arith. mean of mth powers of n quantities and their arith. mean to the mth power.

Maxima and Minima

Let $\varphi(x,y,z)$, $f(x,y,z)$ be any fns of x,y,z. For all values such that:

1) $f = A$
2) $\varphi(x,y,z) \leq f(x,y,z)$

values a,b,c which satisfy 1 and make 2 an equality make $\varphi(a,b,c)$ a max value.
Sym. the values of x,y,z that make $\varphi = A$ and $f \geq \varphi$, the a,b,c as above make f a min value.
This extends to any number of vars and we can state this generally:

Thm. 4.14 Reciprocity Theorem
If for all values x,y,z consistent with $f(x,y,z) = A$, $\varphi(x,y,z)$ have a max value $\varphi(a,b,c) = B$ where B depends on A and if when A increases B also increases and vice versa, then for all values of x,y,z such that $\varphi(x,y,z) = B$, f will have a min value $f(x,y,z) = A$.

From Thm. 4.12. we deduce:

Prop. 4.1. If x,y,z,… be n pos. quantities subject to condition $\sum x = k$ then their product $\prod x$ has a max value $(k/n)^n$ when $x = y = \cdots = k^{1/n}$

Prop. 4.2. If x,y,z,… be n pos. quantities where $\prod x = k$ then $\sum x$ has a min value when $x = y = \cdots = k^{1/n}$

From Thm. 4.12.C1 we get

Prop. 4.3. If x,y,z,… are n pos. quantities where $\sum px = k$ where p,q,r,… are n pos. constants then $\prod x^p$ has a max value $(k/\sum p)^{\Sigma p}$ when $x = y = \cdots = k/\sum p$

Prop. 4.4. x…,p… as above: $\prod x^p = k$ then $\sum px$ is min $(\sum p)k^{1/\Sigma p}$ when $x = y = \cdots = k^{1/\Sigma p}$

From these last two, we deduce:

Prop. 4.5. If λ,μ,…,l,m,…,p,q,… pos. constants and x,y,z,… > 0 then if $\sum \lambda x^l = k$ the $\prod x^p$ is max when $l\lambda x^p/p = m\mu y^q/q = \cdots$

Prop. 4.6. Sym. if $\prod x^p = k$ the $\sum \lambda x^l$ is min when $l\lambda x^l/p = m\mu y^m/q = \cdots$
Cor. 1. If $l = m = \cdots = 1$, $p = q = \cdots = 1$ we get the cases:
1) If $\sum \lambda x = k$ then $\prod x$ max when $\lambda x = \mu y = \cdots$
2) If $\prod x = k$ then $\sum x$ min when $\lambda x = \mu y = \cdots$

Example

A cube is a rectangular parallelepiped of max volume for given surface and min surface for given volume. If we denote the three edges as x,y,z then surface is $2(yz + zx + xy)$ and volume is xyz. If a,b,c = yz,zx,xy then surface is $2(a+b+c)$ and volume is $\sqrt{abc}$. So we want abc max when $\sum a$ given and $\sum a$ min when abc given. This by Prop. 4.1. is done when $a = b = c$ or $yz = zx = xy \therefore x = y = z$.

From Thm. 4.13. we deduce:

Prop. 4.7. If $m \notin (0,1)$ and p,q,r,… pos. constants then for all x,y,z,… > 0: $\sum px = k \Rightarrow \sum px^m$ (m constant) has a min when $x = y = \cdots$. If $m \in (0,1)$ this becomes max.

Prop. 4.8. If m > 1, p,q,r,… pos. constants then $\forall x,y,z,\ldots : \sum px^m = k \Rightarrow \sum px$ max when $x = y = \cdots$ and if m < 1 this becomes min.

Generalizing Prop. 4.7.:

Prop. 4.9. if $m/n \notin (0,1)$ p,q,r,…,λ,μ,… pos constants then $\forall x,y,z,\ldots : \sum \lambda x^n = k$ (n const.) $\Rightarrow \sum px^m$ (m const.) has min if $px^m/\lambda x^n = qy^m/\mu y^n = \cdots$ and if $m/n \in (0,1)$ this becomes max.
Cor. 1. If n = 1 and $\lambda = \mu = \cdots$ and $\sum x = k \Rightarrow \sum px^m$ min or max when $px^{m-1} = qy^{m-1} = \cdots$ as $m \notin \in (0,1)$

More generally:

Prop. 4.10. If p,q,r,… fns of x,y,z,… which are real and pos. for $\forall x,y,z,\ldots \in \mathbf{R}$: $\sum px = k \Rightarrow (\sum px^m)(\sum p)^{m-1}$ min or max when $x = y = \cdots$ as $m \notin \in (0,1)$

Example

If $x^3 + y^4 + z^5 = 3$ then $(x^4 + y^5 + z^6)(x^2 + y^3 + z^4)$ has min for ∀pos. x,y,z, when x = y = z = 1
Follows from Prop. 4.10 if $m = 2$, $p = x^2$, $q = y^3$, $r = z^4$.

Let's look at Grillot's (you know Grillot, right?) use of these propositions. We'll call those points on a fn where f' = 0 the **turning points** and look at them in this fn:

$$u = (ax + p)^l (bx + q)^m (cx + r)^n \qquad [1]$$

where l,m,n > 0. This fn is equivalent to

$$u = (\lambda ax + \lambda p)^l\,(\mu bx + \mu q)^m\,(\nu cx + \nu r)^n \,/\, \lambda^l \mu^m \nu^n \qquad [2]$$

We let

$$l\lambda a + m\mu b + n\nu c = 0 \qquad [3]$$

then

$$l(\lambda ax + \lambda p) + m(\mu bx + \mu q) + n(\nu cx + \nu r) = l\lambda p + m\mu q + n\nu r = k \qquad [4]$$

In all this λ,μ,ν any value and k arbitrary constant.
By Prop. 4.3 $\prod(\lambda ax + \lambda p)$ max when

$$\lambda ax + \lambda p = \mu bx + \mu q = \nu cx + \nu r = k/\sum l \qquad [5]$$

By [3] and [5] we determine x

$$la/(ax + p) + mb/(bx + q) + nc/(cx + r) = 0$$

which quadratic gives x_1, x_2 and [5] gives two values for λ,μ,ν in terms of k: λ_1, λ_2, ...
Then if $\lambda^l, \mu^m, \nu^n > 0$, then u at max turning point and if they are < 0 its a min.

Example

$u = (x + 3)^2(x - 3)$
$u = (\lambda x + 3\lambda)^2(\mu x - 3\mu) / \lambda^2\mu$
$\therefore 2(\lambda x + 3\lambda)(\mu x - 3\mu)$
provided $2\lambda + \mu = 0$ [1]
and $6\lambda - 3\mu = k$ [2]
$\therefore (\lambda x + 3\lambda)^2(\mu x - 3\mu)$ is max if
$\lambda x + 3\lambda = \mu x - 3\mu$ [3]
$\therefore 2/(x+3) + 1/(x-3) = 0$ (by [1])
$\therefore x = 1$
From [2],[3] $\lambda = k/12$ $\mu = -k/6$ $\therefore \lambda^2\mu < 0$
$\therefore$ u min @ x = 1

Note that there is a max above at x = -3 but it eludes this method. This approach can be pursued, using **Purkiss's Theorem**, to determine the turning points of sym. fns of any number of vars. But unless your interest is in sym. fns or lots and lots of vars, these results are far more easily obtained through the Calculus.

Limits

Some basic remarks from what we know out of DME:

1) Let f(x) have the limit l as $x \to a$. If a fn is continuous in the neighborhood of its limit, we can subject the fn to any transformation which is admissible on the hypothesis that the argument of x has any value in the neighbor of critical value a. The transformation must take into account the behavior of f as a is approached from above or below.

2) If as $x \to a$, $f(x) \to l$ then $f(a+h) = l+d$ where d is a fn of a and h and as $h \to 0$, $d \to 0$.

3) Note that any ordinary value of a fn satisfies the definition of a limiting value, i.e. if $f(x) = 3x + 2$ the limit as $x \to 2$ of f is 8. This trivial truth allows the simplification of some proofs.

4) Consider the critical values of u^v. If a>1 and we take log as $\log_a$, this is equivalent to $a^{v \log u}$. So u^v is determinate when $v \log_a u$ is determinate. The cases of indeterminacy are:

1. $v = 0, \log_a u = 0$ or $v = 0$ $u = \infty \Rightarrow 0^\infty$
2. $v = 0, \log_a u = \infty$ or $v = 0$ $u = 0 \Rightarrow 0^0$
3. $v = \pm\infty, \log_a u = 0$ or $v = \pm\infty$ $u = 1 \Rightarrow 1^{\pm\infty}$

All of these depend on $a^{0 \cdot \infty}$ which is $a^{0/0}$ which is a case of the indeterminacy of 0/0.

Recall that the limit of a sum of fns of x is the sum of their limits so long as that sum does not take the form $\pm\infty$. The limit of a product of such fns is the product of their limits so long as the product does not take the form $0 \cdot \infty$. The limit of a quotient of such fns is the quotient of their limits so long as that quotient is not in the form 0/0 or ∞/∞. Where values are not infinite, the above is based on the continuity of the sum, product, or quotient.

Thm. 4.15. A fn F(u,v,w,...) of u,v,w,..., which is determinate and finite in value and continuous when the limits of f(x), φx, χx , ... = u_0, v_0, w_0, ... then the limit of F(f(x), φx, χx, ...) = $F(u_0, v_0, w_0, \ldots)$

Proof

You can prove this by remark #2 above and our definition of a continuous fn.

Example

The limit as x→1 of $((x^2 - 1)/(x - 1))^2$ is the square of the limit as x→1 of $(x^2 - 1)/(x - 1)$ or 4. And the limit as x→1 of the log of this ifrac is the log of the limit of this fn as x→1 or log 2.

Let's look at the forms of 0/0 and ∞/0 with ifracs. If we put num and denom in ascending order and factor out as much x as possible, then when x = 0 the limit as x→0 is finite and not equal to zero if num and denom same degree. The limit is zero if denom is lowest degree and ∞ if num is lowest degree.

Examples

1) $$\frac{2x^2 + 3x^3 + x^4}{3x^2 + x^4 + x^6} \Rightarrow \frac{2 + 3x + x^2}{3 + x^2 + x^4} \quad \text{limit } x \to 0 = \frac{2}{3}$$

You can see that "same degree" means "after x→0".

2) $$\frac{2x^3 + 3x^4 + x^5}{3x^3 + x^4 + x^6} \Rightarrow \frac{2x + 3x^2 + x^3}{3 + x^2 + x^3} \quad \text{limit } x \to 0 = \frac{0}{3} = 0$$

3) $$\frac{2x^4 + x^6}{x^6 + x^8} \Rightarrow \frac{2 + x^2}{x^2 + x^4} \quad \text{limit } x \to 0 = \frac{2}{0} = \infty$$

The form ∞/∞ can only arise in an ifrac when x = ∞. By same method, but noting the highest factor, if same degree the limit is finite, if denom highest the limit is zero, and if num highest the limit is infinite.

Example

$$\frac{3x^3 + x^4}{2x^2 + x^3 + 3x^4} \Rightarrow \frac{3/x + 1}{2/x^2 + 1/x + 3} \Rightarrow \frac{0 + 1}{0 + 0 + 3} \quad \text{limit } x \to 0 = \frac{1}{3}$$

You can easily make up examples for yourself for the other two cases.

If f(x)/φx = 0/0 for x = a ≠ 0 then by the Remainder Theorem (x - a) is a common factor. Removing the common factors, we can determine the limit.

Example

0 for x = 2: $$\frac{3x^4 - 10x^3 + 3x^2 + 12x - 4}{x^4 + 2x^3 - 22x^2 + 32x - 8} \quad \text{nval} = \frac{29416}{10116} = \frac{2^3 \cdot 3677}{2^7 \cdot 79}$$

From the nvals, both have factors of (x - 2). If we divide three times by long division, we arrive at the remainders 15 and 14 and the limit as x→2 is 15/14. Note that the first two divisions have remainder 0. So it turns out that both had factors of $(x - 2)^2$. Had all coeffs been pos., the nval would have revealed they both had factors of $(x - 2)^3$.

Let's evaluate the same ifrac by changing the var. Let x = a+z and evaluate the limit when z = 0 for f(a+z)/φ(a+z):

$$\frac{3(2+z)^4 - 10(2+z)^3 + 3(2+z)^2 + 12(2+z) - 4}{(2+z)^4 + 2(2+z)^3 - 22(2+z)^2 + 32(2+z) - 8}$$

Now we arrange by ascending powers of z and we only need the lowest ones. After expansion, taking all the shortcuts possible (try it and see) we have:

$$\frac{15z^2 + Pz^3 + \cdots}{14z^2 + Qz^3 + \cdots} \Rightarrow \frac{15}{14}$$

To do this, first get all your coeffs of z. Oops, they disappear. Then coeffs of z^2 and we have 15/14 and we're done. Here x = 2 + z, so z = x - 2. But you knew that. All of this works for fractional powers of x:

$$\frac{x^{1/2} + x^{2/3} + 3x^{3/4}}{x^{1/3} + 2x^{1/2} + x} \quad \text{Now divide by the lowest power: } x^{1/3}$$

$$\therefore \quad \frac{x^{1/6} + x^{1/3} + 3x^{5/12}}{1 + 2x^{1/6} + x^{2/3}} \quad \text{limit } x \to 0 = \frac{0}{1} = 0$$

Thm. 4.16. $\forall m \in \mathbf{Q}$, as $x \to 1$ limit $(x^m - 1)/(x - 1) \to m$.

Proof

We use Thm. 4.10. For x>0, $\forall m \neq 0$, $x^m - 1$ is between $mx^{m-1}(x - 1)$ and $m(x - 1)$
$\therefore$ $(x^m - 1)/(x - 1)$ is between mx^{m-1} and m. As $x \to 1$, $mx^{m-1} \to m$. The same is therefore true of $(x^m - 1)/(x - 1)$. It is trapped between mx^{m-1} and m. ■

Example

Evaluate $\log(x^{3/2} - 1) - \log(x^{1/2} - 1)$ when x = 1
Let $L_1 \equiv$ limit $x \to 1$. **Note: we will continue to use this limit notation.**
Now for laziness sake, let's abbreviate L_1 to L

$L(1) = L \log((x^{3/2} - 1)/(x^{1/2} - 1))$
$\quad = \log (L (x^{3/2} - 1)/(x^{1/2} - 1))$ (Thm. 4.16)
$\quad = \log (L (x^{3/2} - 1)/(x - 1) \; / \; L (x^{1/2} - 1)/(x - 1))$
$\quad = \log (3/2 / 1/2) = \log 3$

The following theorem is the basis of differentiation of exponential fns in general. So I'm giving a proof by Fort. (You know Fort, right?)

Thm. 4.17 The limit of $(1 + 1/x)^x$ as $x \to \infty$ is a finite number denoted by ε. (Note that from text to text, this ε notation may be e or anything else the writer can find that's an "e". I will use ε and e indiscriminately. Try to keep up.)

Fort's Proof

$a > b > 0, m > 1 \Rightarrow ma^{m-1}(a - b) > a^m - b^m > mb^{m-1}(a - b)$ [1]

$a = (y + 1)/y,\ b = 1,\ m = y/x: y > x > 1$

$\therefore\ ((y+1)/y)^{y/x} - 1 > 1/x\ \therefore\ (1 + 1/y)^{y/x} > 1 + 1/x$

$\therefore\ (1 + 1/y)^y > (1 + 1/x)^x$ recall $y > x$ [2]

Sym. If $a = 1$ and $b = (y-1)/y$, m,x,y as above: $1/x > 1 - ((y-1)/y)^{y/x}$

$\therefore\ (1 - 1/y)^y > (1 - 1/x)^x$

$\therefore (1 - 1/y)^{-y} < (1 - 1/x)^{-x}$ where $y > x$ [3]

$\therefore x \to +\infty \Rightarrow (1 + 1/x)^x$ increases and $(1 - 1/x)^{-x}$ decreases ([2],[3])

$x^2 > x^2 - 1\ \therefore\ x/(x - 1) > (x + 1)/x\ \therefore\ (1 - 1/x)^{-1} > 1 + 1/x$

$\therefore\ (1 - 1/x)^{-x} > (1 + 1/x)^x$ [4]

$\therefore$ limits of $(1 - 1/x)^{-x}$ and $(1 + 1/x)^x$ cannot pass each other

$x \to +\infty$, $(1 - 1/x)^{-x}$ diminishes to finite limit A, $(1 + 1/x)^x$ increases to finite limit B.

$\therefore$ A = B as $(1 - 1/x)^{-x} - (1 + 1/x)^x = (x/(x-1))^x - ((x+1)/x)^x$ and by [1]:

$1/x(1/(x+1))^x > x/(x-1)^x - ((x+1)/x)^x > 1/(x(1 - 1/x^2)) \cdot ((x+1)/x)^x$ [5]

As $(x/(x-1))^x$ and $((x+1)/x)^x$ remain finite as $x \to \infty$ the upper and lower limits in [5] remain finite as $x \to \infty$.

$\therefore$ middle term remains finite ■

And this finite limit as $x \to \infty$ of $(1 + 1/x)^x$ is Euler's constant ε as follows:

Cor. 1. $L_0(1 + x)^{1/x} = \varepsilon$ as $L_\infty(1 + 1/z)^z = \varepsilon$. Then let $z = 1/x$.

Cor. 2. $L_\infty \log_a(1 + 1/x)^x = L_0 \log_a(1 + x)^{1/x} = \log_a\varepsilon$ which follows from $\log_a x$ being a continuous fn of y for finite y.

Cor. 3. $L_\infty(1 + y/x)^x = L_0(1 + xy)^{1/x} = e^y$ from letting $1/z = y/x$

Cor. 4. $L_0(a^x - 1)/x = \log_a a$

Proof

$y = a^x - 1 \therefore x = \log_a(1 + y)\ \therefore$ as $x \to 0, y \to 0$

$L_{x\to0}(a^x - 1)/x = L_{y\to0}\, y/\log_a(1 + y)$

$= L\ 1/\log_a(1 + y)^{1/y}$

$= 1/\log_a(L(1 + y)^{1/y})$

$= 1/\log_a\varepsilon = \log_a a$ ■

Cor. 5. $x > 0 \Rightarrow e^x > 1+x$ and $\ln(1 + x) < x$

$x \in (0,1) \Rightarrow e^{-x} > 1 - x$ and $-\ln(1 - x) > x$

Proof of this uses $e > (1 + 1/n)^n$ as $n \to \infty \therefore e^x - 1 > (1+1/n)^{nx} - 1 > nx((1+1/n) - 1) > x$

Cor. 6. Let notation of $lx, l^2x, \ldots = \ln x, \ln(\ln x), \ldots, x > 1, r \in \mathbf{N} \Rightarrow$

$1/(xlxl^2x\ldots l^rx) > l^{r+1}(x+1) - l^{r+1}x > 1/((x+1)l(x+1)l^2(x+1)\ldots l^r(x+1)$

Cor. 7. $L_\infty(l^r(x+1) - l^rx) = 0 \quad l_\infty[\ (l^{r+1}(x+1) - l^{r+1}x)xlxl^2x\ldots l^rx\ ] = 1$

I leave the proofs of Cor. 6 and 7 to your curiosity. You are hoping you won't see them again. But you will. I included a lot of the above for two reasons. First, most introductions to ε seem to start in the middle where this one starts at something resembling the beginning. And secondly, you will need to understand why the derivative of a^x is $\ln a \cdot a^x$ and now you do. If you are going in the direction of inequalities the proofs of the next three theorems are worth pursuing for their technique.

Thm. 4.18. For any critical value of x, $L(f(x))^{\varphi x} = (Lf(x))^{(L\varphi x)}$ when RHS determinate.

Many of these proofs use the fact that $u^v = e^{v \ln u}$ as in $(f(x))^{\varphi x} = e^{\varphi x \ln f(X)}$.

Thm. 4.19. Cauchy's Theorem (as if he only had one theorem…)
$L_\infty(f(x+1) - f(x)) = L_\infty(f(x)/x)$ when LHS determinate.

Thm. 4.20. Also Cauchy's Theorem
$L_\infty f(x+1)/f(x) = L_\infty (f(x))^{1/x}$ when LHS determinate.

Cauchy states that these apply to a fn of n such as n! where n takes the place of x.

Thm. 4.21. $a>1 \Rightarrow$ 1) $L_\infty a^x/x = \infty$ 2) $L_\infty \log_a x / x = 0$ and 3) $L_0\, x \log_a x = 0$

Examples

1) Show $a>1, n>0 \Rightarrow L_\infty a^x/x^n = 0$

$L_\infty a^x/n^n = L(a^{x/n}/x)^n = (L\, a^{x/n}/x)^n = \infty^n = \infty$

Note: $a>1, n>0 \therefore a^{1/n} > 1 \therefore L(a^{1/n})^n/x = 0 \;\therefore \infty^n = \infty$

2) $x \in \mathbf{R}$ constant $\Rightarrow L_\infty x^n/n! = 0$

$n \to \infty$, x finite $\therefore \exists k \in \mathbf{N}: x < k < n$

$\therefore x^n/n! = x^{k-1}/(k-1)! \cdot x/k \cdot x/(k+1) \cdots x/n$

$x < k \therefore L_\infty (x/k)^{n-k+1} = 0$

Thm. 4.22. Fundamental Theorem of the Form 0^0
$L_0\, x^x = 1$
Proof
$Lx^x = Le^{x \ln x} = e^{L\, x \ln x} = e^0 = 1$ ■

Examples

1) $L_0(x^n)^x = 1$

$L(x^n)^x = Lx^{nx} = L(x^x)^n = 1$

2) $n>0, y = x^n \Rightarrow L_0 x^y = 1$

$Lx^y = Le^{y \ln x} = e^{L y \ln x} = e^0 = 1$

Thm. 4.23. u,v fns of x, $u(a) = v(a) = 0$, $L_a v/u^n = l$ where n finite in $\mathbf{N} \Rightarrow$
$L_a u^v = 1$ provided the limit is approached such that $u > 0$
Proof
$Lu^v = L(u^{u \wedge n})^{v/u \wedge n}$ (where $u \wedge n \equiv u^n$) $= (Lu^{u \wedge n})^{Lv/u \wedge n}$
$n>0 \;\therefore\; L_0 u^{u \wedge n} = 1 \;\therefore\; Lu^v = 1^l = 1$ ■
Also if $L_a v/u^n = \infty$ the form is 1^∞ and undefined.
And if Lv/u determinate and finite then $Lu^v = 1$
$\therefore Lu^v = 1$ whenever u,v are alg. fns of x.

Example of $u^v = e^{v \ln u}$

Evaluate $x^{1/\ln(e^\wedge x - 1)/x}$ as $x \to 0$

Let this use $L((e^x-1)/x) = 1 \therefore x^{1/\ln(e^\wedge x - 1)} = e^{\ln x/\ln(e^\wedge x - 1)}$

$\ln x/\ln(e^x-1) = \ln x/(\ln((e^x- 1)/x) + \ln x) = 1/((e^x-1)/x \;/\; (\ln x + 1))$

By 4.17.Cor.4, $L \ln((e^x- 1)/x) = 0 \;\; L \ln x = -\infty$

$\therefore \; L (\ln x)/(\ln(e^x- 1)) = 1 \;\therefore\; L\, x^{1/\ln(e^\wedge x - 1)} = e$

The fundamental case for the form 1^∞ is $L_\infty(1 + 1/x)^x = L_0(1 + x)^{1/x} = e$.

Thm. 4.24. u,v fns of x: $x = a \Rightarrow (u = 1) \wedge (v = \infty) \Rightarrow Lu^v = e^{Lv(u - 1)}$
when Lv(u-1) is determinate.

Sums of Infinite Series

Thm. 4.25. $r + 1 > 0 \Rightarrow L_\infty(1^r+ 2^r+ \cdots + n^r)/n^{r+1} = 1/(r + 1)$

Proof

$(r + 1)x^r(x + y) >< x^{r+1}- x^r >< (r +1)y^r(x - y)$

Let $x = p$, $y = p - 1$ then let $x = p + 1$, $y = p$

$\therefore (p + 1)^{r+1}- p^{r+1} >< (r + 1)p^r >< p^{r+1}- (p - 1)^{r-1}$

Let p = 1,2,3,... and add

$\therefore (n + 1)^{r-1}- 1 >< (r + 1)(1^r+ 2^r+ \cdots + n^r) >< n^{r+1}$

$\therefore ((1 + 1/n)^{r+1}- 1/n^{r+1})/(r+1) >< (1^r+ \cdots + n^r)/n^{r+1} >< 1/(r+1)$

$L_\infty(1 + 1/n)^{r+1}= 1$ and $(r+1)>0 \Rightarrow L_\infty 1/n^{r+1}= 0$

$\therefore$ middle term trapped between equal limits

Cor. 1. $s \in \mathbf{N}$, $(r+1)>0 \Rightarrow L_\infty(1^r+ 2^r+ \cdots + (n-s)^r)/n^{r+1} = 1/(r+1)$

Cor. 2. if a constant $\in \mathbf{R}$ $(r+1) > 0 \Rightarrow L_\infty((a+1)^r+ (a+2)^r+ \cdots + (a+n)^r)/n^{r+1}= 1/(r+1)$

Cor. 3. if a,c constant $\in \mathbf{R}$ $(r+1) \neq 0 \Rightarrow$
$L_\infty((na+c)^r+ (na+2c)^r+ \cdots + (na+nc)^r)/n^{r+1} = ((a+c)^{r+1}- a^{r+1})/(c(r+1))$

The next two are by Dirichlet. (And this next little number ...)

Thm. 4.26. $a,b,r > 0 \Rightarrow$ as $n \to \infty$ the $\sum$ of n terms of

$$1/a^{r+1}+ 1/(a+b)^{r+1}+ 1/(a+2b)^{r+1}+ \cdots + 1/(a+nb)^{r+1}$$

is finite for all finite r and if $\sum_{1-\infty} 1/(a+nb)^{r+1}$ denotes this sum then

$$L_{r\to\infty}\, r \cdot \sum_{1-\infty} 1/(a+nb)^{r+1} = 1/b$$

Thm. 4.27. If k_i [1-n]>0 each ≥ its antecedent and $L_\infty T/t = a$ where T is the number of k's that do not exceed t then $\sum_{1-\infty} 1/k_n{}^{r+1}$ is finite for all finite r>0 and

$$L_{r\to 0}\, r \cdot \sum_{1-\infty} 1/kn^{r+1} = a$$

Cor. 1. $1/(r(a-1)^r) > L_{n\to\infty}(1/a^{r+1} + 1/(a+1)^{r+1} + \cdots + 1/(a+n)^{r+1}) > 1/(ra^r)$

Convergence of ∑ and ∏

Some remarks on where DME left us. Infinite series converge, diverge or **oscillate**. Between what values does this series oscillate?

$$3 - 1 - 2 + 3 - 1 - 2 + \cdots$$

Series converge **more or less rapidly**. A geometric series is more rapidly convergent the smaller its common ratio. And the elements of a series are taken in a **given order**. Dirichlet noted that some series might converge to any value or become divergent according to the order of their terms.

Thm. 4.28. Let S_n denote $\sum u_i$ [1-n] and S denote its limit or value. Let R_{nm} denote the sum of the m terms after element u_n. The necessary and sufficient conditions of convergence of S_n is that S_n is finite ∀n and that for large enough n for ∀m,p, $R_{nm} < p$. in other words, no matter how small p is, p is larger than the residue.
Cor. 1. $L_\infty u_n = 0$ as $u_n = S_n - S_{n-1} = R_{(n-1)1} = 0$ **if convergent**.
Cor. 2. Let $R_n = L_\infty R_{nm} \Rightarrow S_n = S - R_n$. Here R_n is the **residue** of the series and R_{nm} is the **partial residue**. $R_n = \sum u_i$ [(n+1)-∞] ∴ the residue of a convergent series is convergent.
Cor. 3. Convergence or divergence is unaffected by ignoring a finite number of terms. Which is to say, infinity is not a number. We are concerned with the implications of the form of the terms and their relations.

Example

$\sum 1/1 \ln(2^2/(1\cdot3)) + 1/2 \ln(3^2/(2\cdot4)) + \cdots + 1/n \ln((n+1)^2/(n(n+2))$
$(n+1)^2/(n(n+2)) = (1+1/n)/(1 + 1/(n+1))$
$\therefore R_{nm} = 1/(n+1) \ln((1 + 1/(n+1))/(1 + 1/(n+2)) + \cdots + 1/(n+m) \ln((1 + 1/(n+m))/(1 + 1/(n+m+1)))$
$< 1/(n+1)(\ln((1+ 1/(n+1))/(1+1/(n+2)) + \cdots + \ln((1+1/(n+m))/((1 + 1/(n+m+1)))$
$< 1/(n+1) (\ln((1+1/(n+m))/((1 + 1/(n+m+1)))$ [1]
$n\to\infty \Rightarrow \forall m, L_\infty R_{nm} \to 0$
If in [1] n = 0 m = ∞ noting $S_n = R_{n0}$ then

$$S_n < \ln ((1 + 1/1)/(1 + 1/(n+1)) < \ln 2$$

and series converges. In [1] set m = ∞ and then the residue is

$$R_n < [\ln(1 + 1/(n+1)]/(n+1)$$

which shows the rapidity of convergence and number of terms needed for a numerical approximation of any given accuracy. Now let $\sum u_n$ denote $\sum u_i$ [1-n] and we have the following theorems which you can prove to yourself by reasoning on S_n, S, R_{nm} and so forth.

Thm. 4.29. $\forall i\ u_i,v_i > 0$, $u_i < v_i$ and $\sum v_n$ convergent $\Rightarrow \sum u_n$ convergent. If $u_i > v_i$ and $\sum v_n$ divergent $\Rightarrow \sum u_n$ divergent.

This last is the opposite of a sum trapped between to equal limits. Here it is either forced by the other to converge or diverge. Now let S refer to u_i and S' to v_i and so on.

Thm. 4.30. $\forall i\ v_i>0 \quad u_i/v_i$ finite $\Rightarrow \sum u_n$ converges or diverges as $\sum v_n$ converges or diverges.

Proof

Consider R_n and u_i/v_i [n+1 - n+m] and A,B smallest and largest of $u_i/v_i \Rightarrow$

$$A < \sum u_n/\sum v_n < B$$

Each fraction finite ∴ A,B finite ∴ $R_{nm} = CR'_{nm}$ where C finite value dependent on n,m. ∴ $S_n = R_{n0}$ finite or infinite as S'_{nm} finite or infinite. ■

Recall the ratio of the consecutive terms in a series in DME, this is now **ratio of convergence** (roc).

Thm. 4.31. $\forall i\ u_i,v_i > 0$, $u_{n+1}/u_n < v_{n+1}/v_n$ and $\sum v_n$ convergent $\Rightarrow \sum u_n$ convergent. If $u_{n+1}/u_n > v_{n+1}/v_n$ and $\sum v_n$ divergent $\Rightarrow \sum u_n$ divergent. (And of course ∀i means any i after some equal finite number of terms.)

Thm. 4.32. If a series with neg. terms converges when the neg. terms are made pos. then the original series also converges.

Def. A series that converges when all its terms are taken as positive is **absolutely convergent**.

Convergence Tests

These apply to series where after r terms the terms are only positive.

Test I: $\sum u_n$ converges or diverges as $u_n^{1/n}$ is ultimately <> 1. If this equals one then the test is meaningless.

Example

$\sum 1/(1 + 1/n)^{n^2} \Rightarrow L_\infty u_n^{1/n} = 1/(L(1 + 1/n)^n = 1/e$
$2 < e < 3 \therefore 1/e < 1 \therefore$ convergent

Test II: $\sum u_n$ converges or diverges as its roc ultimately <>1. Again, no result if roc = 1.

You can see that "ultimately" means the L_∞ of a thing. Note that Test I ≡ Test II by Cauchy's theorem above where

$$L_\infty u_{n+1}/u_n = L_\infty u_n^{1/n}$$

which you may only now interpret in this way. We can also note that if φn is alg. fn of n then $\sum\varphi(n)x^n$ converges or diverges as x <> 1 and that $\sum x^n/n!$ is convergent ∀x. The next theorem is one of those things that actually seem cool to me. So I will subject you to the full proof. It also has a bunch of corollaries with proofs. Sorry.

Thm. 4.33. Cauchy's Condensation Test

∀n f(n)>0, a ≥ 2, and f monotonically decreases as n increases then ∑f(n) converges or diverges as $\sum a^n f(a^n)$ converges or diverges.

Proof

$$\begin{aligned}\sum f(n) = &[f(1) + \cdots + f(a - 1)]\\ &+ (f(a) + \cdots + f(a^2- 1))\\ &+ (f(a^2) + \cdots + f(a^3- 1))\\ &+ \cdots\\ &+ (f(a^m) + f(a^m+ 1) + \cdots + f(a^{m+1}- 1)) + \cdots \qquad [1]\end{aligned}$$

We can neglect any a-1 terms ∴ ∑f(n) converges or diverges (cnv/div) as [1] cnv/div. Because [1] is mono-decreasing

$$(a^{m+1}- 1)f(a^m) > [1] > ((a-1)/a)a^{m+1}f(a^{m+1}) \qquad [2]$$

∴ By Test I [1] converges if ∑LHS[2] converges and diverges if ∑RHS[2] diverges.

By Test II ∑LHT[2] converges if $\sum a^m f(a^m)$ converges and ∑RHS[2] diverges is $\sum a^{m+1}f(a^{m+1})$ diverges.

And $\sum a^m f(a^m) \equiv \sum a^{m+1}f(a^{m+1}) = \sum a^n f(a^n)$ ■

Keep in mind here that f(n) is pos. and mono-decreasing after r terms. Also guess why a needs to be ≥ 2. Get used to ε being everywhere. It's like a bad penny. The following corollaries are post-Cauchy.

Cor. 1. Thm. 4.33. holds if 0 < a < 2. Or let's just say a needs to be positive.

Cor. 2. ∑f(n) cnv/div as $\sum e n e^2 n e^3 n \cdots e^r n f(e^r n)$ cnv/div

The proof of Cor. 2. comes from repeated application of 4.33. on a = ε. We are leading up to the Logarithmic Scale of Convergence in the next two corollaries. This was developed by De Morgan and if you want to see it in use and to see the world's most amazing exponents of ε, check out De Morgan's text on double algebra, which he also invented. We'll be using the l^i notation where $l^2 \equiv \ln(\ln(\text{something})$. Then we'll give a taste of the scale's use, in no way comparable to De Morgan's, before we go on. Even if you don't focus on this enough to load it into your head, it's worth knowing it's there.

Cor. 3. ∑f(n) cnv/div as the first of these fns which do not vanish as x→∞ has pos. or neg. limit:

$$T_0 = \ln f(x) / x$$
$$T_1 = \ln (xf(x)) / \ln x$$
$$T_2 = \ln(xl^1xf(x)) / l^2x$$
$$\cdots$$
$$T_r = \ln(xlxl^2x\cdots l^{r-1}xf(x)) / l^rx$$

[proof follows]

Proof

By Cor. 2., $\sum f(n)$ cnv/div as that $\sum$e-thing cnv/div. And that latter series cnv/div as $L_\infty(ene^2ne^3n\cdots e^rnf(e^rn))^{1/n}$ <> 1. And this is equivalent to

$\equiv L_\infty \log_a(en\cdots f(e^rn))$ <> 0

$\equiv L_\infty(\log_a(en\cdots f(e^rn))/n$ <> 0

Now let a = e. If $x = e^rn$ then $lx = e^{r-1}n \ldots l^rx = n$ and $n\to\infty \Rightarrow x\to\infty$ ∴

$\equiv L_\infty l(xlxl^2x\cdots l^{r-1}xf(x))/l^rx$ <> 0

Cor. 4. Each of these series converges if $\alpha > 0$ and diverges if $\alpha \le 0$:

$\sum 1/n^{1+\alpha}$ [1]

$\sum 1/(n(l^1n)^{1+\alpha}$ [2]

$\sum 1/nln(l^2n)^{1+\alpha}$ [3]

. . .

$\sum 1/nlnl^2n\cdots l^{r-1}n(l^rn)^{1+\alpha}$ [r+1]

Proof

Let $P_r(n)$ denote $nlnl^2n\cdots l^rn$: $P_0(n) = n$, $P_1(n) = nln$, …. By Cauchy's Condensation Test, [1] converges as $\sum a^n/(a^n)^{1+\alpha} \cong \sum(1/a^\alpha)^n$ converges where $\cong$ just means LHS converges if RHS does. And RHS is G.P. with common ratio $1/a^n$ and converges if a > 0. So [2] converges by the same law as $\sum a^n/a^n(la^n)^{1+\alpha} = \sum(la)^{1+\alpha}n^{1+\alpha} \cong 1/n^{1+\alpha}$ ∴ [2] converges. Assume [r] converges.

Cnv of [r+1] $\sum a^n/a^nla^nl^2a^n\cdots l^{r-1}a^n(l^ra^n)^{1+\alpha} \cong \sum 1/nlal(nla)\cdots l^{r-2}(nla)(l^{r-1}(nla))^{1+\alpha}$

$\alpha>0$ $a>e \Rightarrow la > 1$ $nla > n$

∴ $1/nlal(nla)\cdots l^{r-2}(nla)(l^{r-1}(nla))^{1+\alpha} < 1/nln\cdots l^{r-2}n(l^{r-1}n)^{1+\alpha}$

$\alpha > 1 \Rightarrow \sum 1/P_r(n)(l^{r-1}n)^\alpha$ converges [A]

∴ $\sum 1/P_n(n)(l^rn)^\alpha$ converges [B]

Let $\alpha \le 0$ $2 < a < e \Rightarrow nla < n$ and B more divergent than A. ■

Logarithmic Scale of Convergence

Let's bring this idea to a closure, even though it is a notational nightmare. These series just above [1] - [r+1] form a descending scale, allowing us to compare series whose roc goes to unity. The least convergent of the convergent series of the rth order is more convergent that the most convergent of r+1. Consider nth terms, u_n, u_n' of the rth, (r+1)th series:

$$u_n'/u_n = (l^{r-1}n)^a/(l^rn)^{1+a'}$$

where a is small but positive and a' very large. If $l^{r-1}n = x$ then

$$L_\infty\, u_n'/u_n = L_\infty((x^{n/1+a'})/(lx))^{1+a'} = \infty$$

This next corollary to our ongoing idea is De Morgan's:

Cor. 5. Let $p_x = f(x+1)/f(x)$, If $f(x)>0$ when x is greater than a certain finite value, $\sum f(n)$ is cnv/div as the first of these fns whose limit is finite as $x\to\infty$ is neg/pos.

$$\tau_0 = p_x - 1$$
$$\tau_0 = P_0(x+1)p_x - P_0(x)$$
$$\tau_2 = P_1(x+1)p_x - P_1(x)$$
$$\cdots$$
$$\tau_r = P_{r-1}(x+1) - P_{r-1}(x)$$

Let's look at the power of these ideas.

Example

Determine the convergence of $\varepsilon^{(1 - 1/2 - \cdots - 1/n)/n\wedge r}$

$$\tau_0 = l(f(n)/n = -(1 + 1/2 + \cdots + 1/n + rln))/ln$$
$$1 + (r+1)ln > 1 + 1/2 + \cdots + 1/n + rln > rln + l(n+1) \therefore L = 0$$
$$\tau_1 = l(nf(n))/ln$$
$$= -(1 + 1/2 + \cdots + 1/n + (r-1)ln)/ln$$
$$= -(1 + 1/2 + \cdots + 1/n)/ln - (r-1)$$
$$= L(1 + 1/2 + \cdots + 1/n)/ln = 1 \therefore L\tau_1 = -1 - r + 1 = -r$$
$\therefore$ cnv/div as r >< 0
$$r = 0 \Rightarrow L\tau_0 = L\tau_1 = 0$$
$$\tau_2 = l(nlnf(n)/l^2n = 1 - (1 + 1/2 + \cdots + 1/n - ln)/l^2n$$
$n\to\infty$ RHT denom$\to\varepsilon$ $\therefore$ $L\tau_2 = 1 > 0$ $\therefore$ divergent

You can see the power of this as well as how much you must master to use it. We now give the l^n notation a rest and consider series with infinitely many negative terms which may not be abs. conv. We determine convergency by associating each neg. term with a preceding or following pos. term The the terms generally become all pos. or all neg. If the terms of this equivalent series go to zero then any difference between the original and the rearranged series goes to zero too. Convenient. All this produces a false result if used on an oscillating series, of course.

Example

$1/1 - 1/2 - 1/3 + 1/4 - 1/5 - 1/6 + \cdots + 1/3n-2 - 1/3n-1 - 1/3n + \cdots$ [1]
Compare to
$1/1 - (1/2 + 1/3) + 1/4 - (1/5 + 1/6) + \cdots + 1/3n-2 - (1/3n-1 + 1/3n)$ [2]
The difference here is the (2n-1)th term.
If $S_n, S'_n = \sum$(n terms) or [1],[2] $\Rightarrow S_{3n-2} = S'_{2n-1}$ $S_{3n-1} = S'_{2n-1} - 1/(3n-1)$ $S_{3n} = S_{2n}$
$L1/3n-1 = 0 \therefore \forall n, LS_n = LS'_n$ $\therefore$ [1] cnv/div as [2] cnv/div. That [1] cnv is shown by comparing it to $\sum(1/(3n-2) - 1/(3n-1) - 1/3n)$ [3]
Using S" for this one we can show $LS''_n = LS_n$. But nth term of [3] is

$$(-9 + 12/n - 2/n^2)/((3-2/n)(3-1/n)(1/3n))$$

and has a ratio with $\sum 1/n$ which is always finite.
But $\sum 1/n$ diverges $\therefore$ [3] diverges $\therefore$ [1] diverges.

Thm. 4.34. Rule of Semi-Convergence
$\forall n$, $(u_n > 0) \wedge (u_n > u_{n+1}) \Rightarrow u_1 - u_2 + u_3 - \cdots + (-1)^{n-1}u_n + (-1)^n u_{n+1} + \cdots$ converges or oscillates as $L_\infty u_n =/\neq 0$.
Cor. 1. With u_i as above, this converges: $(u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{m-1} - u_m) + \cdots$

We consider series and the laws of algebra:

Associative Law
If a series is convergent, association produces no affect on an infinite sum. Associative law holds for a convergent series.

Commutative Law
Commutation can only be applied to an absolutely convergent series. It cannot be applied to semi-convergent series and commutation will produce false results.

Addition
If $\sum u_n$, $\sum v_n$ converge to S,T then $\sum(u_n + v_n)$ converges to S+T

Distributive Law
1. $a \in \mathbf{R}$ $\sum u_n \rightarrow S \Rightarrow \sum a u_n \rightarrow aS$
2. $\sum u_n$, $\sum v_n \rightarrow S,T$ and at least one of them is abs. cnv. $\Rightarrow$
   $u_1 v_1 + (u_1 v_2 + u_2 v_1) + \cdots + (u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1) + \cdots$
   converges to ST. But if both are semi-cnv the multiplication may fail.

Power Series

Power series ($\sum a_n x^n$) are viewed as complex series here. Note that $\sum x_n + i y_n$ converges if both $\sum x_n$ and $\sum y_n$ converge. $x_n + i y_n = z_n$ and p_n denotes the modulus of z_n. θ_n denotes $\arg(z_n)$. Then $\sum z_n$ converges if $\sum p_n$ converges and when this is the case $\sum z_n$ is abs.cnv. Convergence of $\sum p_n$ is sufficient ($\Rightarrow$) but not necessary ($\Leftrightarrow$). Here $z_n = \rho_n(\cos\theta_n + i \cdot \sin\theta_n)$. Now consider $a_n = r_n(\cos\alpha_n + i\sin\alpha_n)$ where r_n, a_n are various f(n) and $x = \rho(\cos\theta + i\sin\theta)$ where ρ,θ are independent of n.

Thm. 4.35. $\sum a_n x^n$ converges if mod x < L(mod a_n/mod a_{n+1})
Proof
The series of moduli is $\sum r_n \rho^n$ which converges if $L(\rho^{n+1} r_{n+1}/\rho^n r_n)$ or $L(r_{n+1}/r_n) < 1$ which is to say $\rho < L(r_n/r_{n+1})$ ■

Three cases arise as $L(r_n/r_{n+1})$ is **1)** = 0 or **2)** = some finite r < $\mathbf{R}^+$ or **3)** = ∞.
1. series converges only if x = 0
2. series converges only if x is inside circle on the origin of radius r. We then have the **circle of convergence** and the **radius of convergence**
   Example: $\sum x^n/n$ converges and r = 1
3. convergent for $\forall x$. Example: $\sum a_n/n!$

Thm. 4.36. If $\sum a_n x^n$ is abs.cnv. when mod x = R' it will be abs.cnv. when x < R'.

If the nth term of series f(n,x), a single-valued fn of x for $\forall n \in \mathbf{N}$ then $\sum f(n,x)$ will, if convergent, be a single-valued, finite fn of x, say φx, and φx is not necessarily continuous. If we have $\sum f(n,x+h)$ and $\sum f(n,x)$ both convergent, then $\sum f(n,x+h) - \sum f(n,x)$ is convergent but its limit is not necessarily zero for any x. The discontinuity of this series, say discontinuous at 0, has a residue $R_n = 1 - S_n = 1/(nx+1)$ when $x \neq 0$. As n increases, $R_n <$ some finite α. But the smaller x is, the larger n must be for $R_n < \alpha$. When x is var, there is no finite limit v for n: $n>v \Rightarrow R_n<\alpha$. This case makes a series **non -uniformly convergent**. And if for some x, like our x=0 here, the limit of n for $v\to\infty$ then series **converges infinitely slowly**. It follows:

Def. (Du Bois-Reymond) If, for values of x within a given region of the complex plane, for $\forall\alpha,n$ $\exists$upper limit v ind. of x: $n > v \Rightarrow R_n <$ mod x. In this case, $\sum f(n,x)$ is **uniformly convergent** within this region.

Thm. 4.37. If $\sum f(n,x)$ uniformly converges then its φx is continuous.

There are two important kinds of power series:

1. a_n ind. of x $\therefore \sum a_n x^n$ as a fn of $x \equiv \varphi x$
2. a_n fn of, say, $n,y \equiv f(n,y)$ where x is considered constant.
   $\therefore \sum f(n,y)$ as fn of $y \equiv \psi y$

Thm. 4.38. Du Bois-Reymond
μ_n ind. x, $w_n(z)$ single-valued fn of n and z and is finite $\forall n$ and finite and continuous for $\forall z\in[a,b]$. Then if $\sum\mu_n$ abs.cnv., $\sum\mu_n w_n(z)$ is a continuous fn of z on [a,b].
Proof
Let $S_n(z)$ be the sum of n terms of $\mu_i w_i(z)$ [1-n], $\forall n$, $\mu_n > 0$.
Let $\Delta w_p = w_p(z+h) - w_p(z)$: $\forall p$, $L_{h\to 0}w_p = 0$
$\therefore S_n(z+h) - S_n(z) = \mu_1\Delta w_1 + \mu_2\Delta w_2 + \cdots + \mu_m\Delta w_m$
$+ \mu_{m+1}w_{m+1}(z+h) + \mu_{m+2}w_{m+1}(z+h) + \cdots + \mu_n w_n(z+h)$
$- \mu_{m+1}w_{m+1}(z) - \mu_{m+2}w_{m+2}(z) - \cdots - \mu_n w_n(z)$
Let ΔW_m, W'_{mn}, W_{mn} be the means Δw_i, $w_{m+i}(z+h)$, $w_{m+i}(z)$
Then, with S', R_{mn} as usual, $S_n(z+h) - S_n(z) = \Delta W_m S'_m + (W'_{mn} - W_{mn})R'_{m|n-m}$
$\forall m,n$, as $n\to\infty$, $W_{m\infty}$, $W'_{m\infty} \to$ finite values by hyp.
$\therefore S_n(z+h) - S_n(z) = \Delta W_m S'_m + (W'_{m\infty} - W_{m\infty})R'_m$
We need to show LHS$\to$0. When h=0, $\Delta w_i \to 0$ because $w_i(z)$ is continuous. Then as S'_m finite and $\sum\mu_m$ convergent $\Rightarrow S_\infty(z\pm 0) - S_\infty(z) = L_{h\to 0}(W'_{m\infty} - W_{m\infty})R'_m$
On RHS, LHT may not go to zero but $R'_m \to 0$ as the residue vanishes. ■

Example

$\sum f(n,x)$ where $f(n,x) = x/((nx+1)(nx-x+1))$
The nth term can be written
$(1/n^2)(x/((x + 1/n)(x - x/n + 1/n))$
$\therefore \mu_n = 1/n^2$ $w_n x$ = RHT and condition of thm. fulfilled when $x\neq 0$.
For $w_\infty(x) = 1/x$ which is finite for $x\neq 0$.
So this series is a continuous fn of $\forall x\neq 0$.

Cor. 1. If power series $\sum a_n x^n$ is abs.cnv. when mod x = R then for ∀x<R, $\sum a_n x^n$ is a continuous fn of x.

Cor. 2. If power series $\sum f(n,y)x^n$ is convergent when mod x < R and R<1 and ∀n, f(n,y) is a single-valued fn continuous on [a,b] then for y from a to b ψy is a cont. fn of y when mod x ≤ R.

Thm. 4.39. Abel's Theorem

If series $\sum a_n$ converges and $\sum a_n x^n$ converges on [0,1) then $L_{x:1\to 0} a_n x^n = \sum a_n$

Note: This asserts convergence of $\sum a_n x^n$, for x∈**R** is continuous inside and on its circle of convergence. This holds for semi-convergent series if you don't alter the order of its terms.

Cor. 1. If $\sum u_n, \sum v_n \to u,v \Rightarrow$ if their product (see distributive law above) converges, it converges to uv and this holds if any or all are semi-convergent.

Thm. 4.40 Principle of Indeterminate Coefficients

If for ∀x, $\sum a_n xn$ converges when mod x < R and if for such x, $a_0+\sum a_n x^n = 0 \Rightarrow a_i\ [1\text{-}n] = 0$.

Cor. 1. If ∀x, mod x < R, $a_0+\sum a_n x^n = b_0+\sum b_n x^n$ and both series converge ⇒ ∀i, $a_i = b_i$

Infinite Products

The value of an infinite product, factors restricted to the form $(1 + u_n)$, is determined in the same way as in the Binomial Theorem

$$(1 + u_1)(1 + u_2)\cdots(1 + u_n)$$

and is the limit of this product as n→∞ denoted mostly $\prod_n$ and sometimes P_n. The former more indicates the process and the latter the value. But you will have to determine, in context, which is actually meant. (You can do it.) If $Lu_n>1$ then $LP_n = 0 \vee \infty$ and such cases are unimportant. So we assume $Lu_n \in (0,1)$. And like ∑, in ∏ we can ignore any finite series of terms. We have four cases:

1) $LP_n = 0$ - convergent but ignored
2) LP_n finite, denoted P - convergent
3) LP_n infinite - divergent
4) LP_n indeterminate, taking one or another of a set of values - analogous to an oscillating ∑.

In we consider $\ln(P_n)$ we reduce the theory of ∏ to the theory of ∑, which is another thing which strikes me as simply elegant, hence cool.

$$\ln P_n = \ln(1 + u_1) + \ln(1 + u_2) + \cdots + \ln(1 + u_n) = \sum \ln(1 + u_n)$$

That **is** cool, isn't it? Let's relate the two ideas:

1) $\sum\ln(1 + u_n)$ diverges, limit = $-\infty \Leftrightarrow \prod(1 + u_n) = 0$
2) $\sum\ln(1 + u_n) \Rightarrow \prod(1 + u_n)$ converges
3) $\sum\ln(1 + u_n)$ diverges, limit = $+\infty \Rightarrow \prod(1 + u_n)$ diverges
4) $\sum\ln(1 + u_n)$ oscillates $\Rightarrow \prod(1 + u_n)$ oscillates

If we restrict u_n to one sign:

1) $Lu_n < 0 \Rightarrow \sum\ln(1 + u_n) = -\infty \wedge \prod(1 + u_n) = 0$
2) $Lu_n > 0 \Rightarrow \sum\ln(1 + u_n) = +\infty \wedge \prod(1 + u_n)$ diverges
$\therefore Lu_n = 0 \Rightarrow \prod(1 + u_n)$ converges (necessary but not sufficient condition)

Here comes Euler again. If $Lu_n= 0 \Rightarrow \prod(1 + u_n)^{1/u\vee n} = \varepsilon$ where like u^n being u^n, u∨n is u_n. (This is a word-processor, not a typesetting shop.) It follows:

1) $L\ln(1 + u_n)/u_n = 1$
2) $\sum\ln(1 + u_n)$ cnv/div as $\sum u_n$ cnv/div

Also, if u_n is ultimately made up of all pos/neg terms, the infinite limits of $\sum u_n$ and $\sum\ln(1 + u_n)$ will be correspondingly pos/neg. So if terms of $\sum u_n$ ultimately have same sign:

1) $\prod(1 + u_n)$ converges $\Leftrightarrow \sum u_n$ converges
2) $\prod(1 + u_n) = 0 \Leftrightarrow \sum u_n = -\infty$
3) $\prod(1 + u_n)$ diverges$\rightarrow+\infty \Leftrightarrow \sum u_n$ diverges$\rightarrow+\infty$

$\prod_n$ is **absolutely convergent** when its sign is ultimately invariable.

Thm. 4.40. $\prod(1 + u_n)$ abs.cnv. $\Leftrightarrow \sum u_n$ abs.cnv.
Cor. 1. If one of $\prod(1 + u_n)$ and $\prod(1 - u_n)$ abs.cnv. then so is the other.

We can talk about residues of $\prod$ just as with $\sum$.

$\prod(1 + u_n)$ cnv. $\Leftrightarrow \forall n$, ($P_n$ finite) $\wedge$ ($L_\infty P_{n+m} - P_n = 0$)

and the latter is equivalent to $L(P_{n+m}/(P_n - 1)) = 0$ or $L(P_{n+m}/P_n) = 1$ and our residue Q is $(1 + u_{n+1})\cdots(1 + u_{n+m})$. So we have proven:

Thm. 4.41. $\prod(1 + u_n)$ cnv $\Rightarrow \forall n$, (P_n finite) $\wedge$ ($L_\infty Q_{mn} = 1$)

Now let u_n have terms in **C**. The two conditions on Thm. 4.41. RHS become mod P_n finite for all n and $L(Q_{mn} - 1) = 0$. For $\prod$ in **C**, $\prod(1 + u_n)$ is convergent if $\prod(1 + \text{mod } u_n)$ is convergent but not conversely.

Def. If $\prod(1 + \text{mod } u_n)$ converges then $\prod(1 + u_n)$ is **absolutely convergent**. If $\prod(1 + u_n)$ converges and $\prod(1 + \text{mod } u_n)$ does not, then $\prod(1 + u_n)$ is **semi-convergent**.

Thm. 4.42. If $\sum(\text{mod } u_n)$ convergent $\Leftrightarrow \prod(1 + u_n)$ abs.cnv.
Cor. 1. $\sum u_n$ convergent $\Rightarrow \prod(1 + u_n x)$ is abs.cnv. where x is ind. of n or x is a fn of n: Lmod $x \neq \infty$ when $n\rightarrow\infty$.

Examples

1) $\prod(1 - x^n/n)$ abs.cnv for ∀x: mod x < 1 (but not when x = 1)

2) $\prod(1 - x/n^2)$ here x is ind. of n ∴ abs.cnv.

Again, the laws of algebra:

1) **Associative Law** requires $Lu_n = 0$ to operate on factors of $\prod(1 + u_n)$
2) **Commutation** holds if $\prod(1 + u_n)$ abs.cnv. but not generally otherwise.
3) If $\prod(1 + u_n)$ and $\prod(1 + u'_n)$ abs.cnv. ⇒
    1) $\prod((1 + u_n)(1 + u'_n))$ converges to the product of their limits
    2) $\prod((1 + u_n)/(1 + u'_n))$ converges to the quotient of their limits so long as no factors vanish in denom.

For **4)** we have the following theorem:

Thm. 4.43. Since $\sum \ln(1 + \mu_n w_n(z)) = \sum \mu_n w_n(z) \ln(1 + \mu_n w_n(z))^{1/\mu \vee n \cdot w \vee n(z)}$ ∴ μ_n and $w_n(z)$ satisfy Du Bois-Reymond's conditions and $\prod(1 + \mu_n w_n(z)$ is a continuous fn of z on [a,b].

Cor. 1. If $\sum a_n x^n$ converges when mod x = R then $\prod(1 + a_n x^n)$ converges to φx, a continuous, finite fn of x for ∀x: mod x ≤ R.

Cor. 2. If f(n,y) is finite, single-valued wrt n and finite, single-valued, and continuous for y∈[a,b] and $\sum f(n,y)x^n$ abs.cnv. when mod x < R, and R < 1 then $\prod(1 + f(n,x)x^n)$ converges to ψx, a finite, single-valued, continuous fn of x for all finite x.

Cor. 3. If $\sum a_n$ abs.cnv. ⇒ $\prod(1 + a_n x)$ converges to ψx just as above.

5) If for a continuum of values of x including 0, $\prod(1 + a_n x^n)$ and $\sum(1 + b_n x^n)$ both abs.cnv. and $\prod(1 + a_n x^n) = \prod(1 + b_n x^n)$ ⇒ ∀i, $a_i = b_i$.

We can extend factorization to $\prod_n$ which holds for abs.cnv. only:

Thm. 4.44. If ∀x, ψx = $\prod(1 + a_n x)$ convergent where Lmod $P_n \neq \infty$ and where as n→∞ for ∀m, L(mod(Q_{mn} - 1) ⇒ ψx = 0 and if ∀i, x has the value $-1/a_i$ and ψx=0 ⇒ for some i, x has the value $1/a_i$.

Cor. 1. If x is on a continuum, denoted (x) including all values $1/a_i$, $1/b_i$ [1-n] and if $\prod(1 + a_n x)^{\mu \vee n}$ and $\prod(1 + b_n x)^{\nu \vee n}$ abs.cnv. on (x) and if f(x),g(x) become neither 0 or ∞ for ∀i, $1/a_i$,$1/b_i$ and if for all x on (x)

$$f(x)\prod(1 + a_n x)^{\mu \vee n} = g(x)\prod(1 + b_n x)^{\nu \vee n}$$

then must each factor of each be raised to the same power and for ∀x ∈ (x), f(x) = g(x)

Cor. 2. It follows that any f(x) which vanishes for any of the $1/a_i$ and for no others outside of the $1/a_i$ then this f(x) can be expressed **uniquely** as a convergent infinite product $f(x)\prod(1 + a_n x)^{\mu \vee n}$, if at all.

Double Series

We get a double series $\sum u_{n,m}$ when two indices n,m ∈ **N** start at 1 and run off to infinity. Conceptually, there are four ways to "take" this sum.

First Way

Conceived as an m row by n col matrix. We let m or n run to infinity and then the other and if $S_n = S_m$ we say $\sum u_{m,n}$ converges to $S_{m,n} = S$ **in the first way**. and if we don't get equal limits we say the series is **nonconvergent** in the first way.

Second Way

We sum m rows to n elements using the same matrix idea. The limits of these are T_i [1-m]. If these limits are all finite then $\sum u_{m,n} = \sum T_i = S'_{m,n}$ **in the second way** as m→∞.

Third Way

Just like the second way but using columns U_i so we get a limit $S''_{m,n} = \sum U_i$
So long as m,n finite we have $S_{m,n} = S'_{m,n} = S''_{m,n}$ but when m,n→∞, we have

$$S' = L_{m\to\infty}(L_{n\to\infty}(S_{m,n})) \text{ and } S' = L_{n\to\infty}(L_{m\to\infty}(S_{m,n}))$$

and S' = S" requires the two ways to lead to the same result.

Fourth Way

Sum the terms, beginning with matrix entry a_{11}, on the diagonal: a_{11}, $(a_{12}+a_{21})$, ... and let that go to infinity. (I'll wait while you try it....) While m,n finite, this never equals the previous methods. And the limits of the first three ways can be equal when this one is infinite.

Now, as before, consider all terms having ultimately the same sign so that we can ignore finitely many consecutive entries in the matrix. Then, as $L(S_{m+p,n+q} - S_{m,n}) = S - S = 0$ as m→∞ for ∀p,q then, as before, any "initial" terms do not affect the limit. And we have an analogous two-diminsional shrinking residue and this is true of all four ways of taking the double limit.

Thm. 4.45. If all the terms of $\sum u_{m,n}$ are > 0 and if the series converges in the first way, then the horizontal, vertical, and diagonal series are convergent and the series converges in all four ways to the same limit.

We define **Restriction A** (RA) for commuting: If term $u_{m,n}$ becomes $u_{m',n'}$ where m' = f(m,n) n' = g(m,n) are both, for and finite m,n, single-valued finite fns, convergence is unaffected. This conserves our vanishing residue.

Cor. 1. If $\sum u_{m,n}$, as series of pos. terms, converges in the first way, commutation of terms under RA leaves Thm. 4.45. intact.

Cor. 2. If all pos. terms of a convergent single series $\sum u_n$ can be arranged into a double series $\sum u_{m',n'}$ where m',n' are fns of n under RA then $\sum u_{m',n'}$ converges in all four ways to the limit of $\sum u_n$.

Thm. 4.46. If a double series of pos. terms converges in any of the four ways to limit S, it converges in all four ways to S and the subsidiary (horiz., vert., diag.) series are all convergent.
Cor. 1. Any series $\sum u_{n'}$ selected from $\sum u_{m,n}$ under RA is convergent if $\sum u_{m,n}$ converges.

Def. If any $\sum u_{m,n}$ converges when all its terms are taken pos. it is **absolutely convergent** (abs.cnv.)

Note that all the above concerning single-sign termed series converging are true of abs.cnv. series as the neg. terms only reduce the residue.

Thm. 4.47. Cauchy's Absolute Convergence Test
If $u'_{m,n}$ are the numerical or pos. values of $u_{m.n}$ (absolute values $|u_{m,n}|$) and if all the horizontal, vertical, and diagonal series of $\sum u'_{m,n}$ converge and the sum of their sums to infinity converge ⇒
1) The horizontal series of $\sum u_{m,n}$ are absolutely convergent and the sum of their sums to infinity converges to some finite S;
2) $\sum u_{m,n}$ converges in the first way;
3) All vertical series are also abs.cnv. and the sum of their infinite sums is some finite S;
4) The same is true of the diagonal sums; and
5) Any series in forms of terms of $\sum u_{m,n}$ under RA is abs.cnv.
Note: It seems to me that Cauchy could state this a little clearer. 1-4 say that $\sum u_{m,n}$ converges in the first three ways and that in all these ways the subordinate series are abs.cnv. to some finite S.
Cor. 1. If $\sum u_n,\sum v_n$ abs.cnv. to u,v ⇒ $\sum(u_nv_1+u_{n-1}v_2+\cdots+u_1v_n)$ abs.cnv. to uv.

It is easy to construct double series where horiz. and vert. series are abs.cnv. but do not have a limit in the first way and have different limits in the second and third ways.

Example
If the first way's limit of $S_{m,n}$ is A+f(m,n) where A is ind. of m,n it is **easy to see** that

$$u_{m,n} = f(m,n) - f(m-1,n) - f(m,n-1) + f(m-1,n-1)$$

So we have only to give f(m,n) a form where

$$L_{m=\infty}(L_{n=\infty}f(m,n)) \neq L_{n=\infty}(L_{m=\infty}f(m,n))$$

so that the series has different limits in 2d and 3d way and no limit 1st way.
Let f(m,n) = (m+1)/(m+n+2) then

$$\begin{aligned} u_{m,n} &= (m+1)/(m+n+2) - m/(m+n+1) - (m+1)/(m+n+1) + m/(m+n) \\ &= (m-n)/(m+n)(m+n+1)(m+n+2) \end{aligned}$$

It is at once **obvious** that the limits of the 2d, 3d, 4th ways are different, for in the first place we observe that $u_{m,n} = -u_{n,m}$. Hence there is a skew arrangement of terms in the array. Therefore each of the diagonal sums is zero and the 4th way limit is 0. Also, due to the arrangement of signs, $T_m = -U_m \therefore S' = -S''$. Chrystal goes on to show that the second and third limits are -½ and ½ and that by noticing that after m=n the terms are negative, therefore $T'_n = \sum_{n=1\to m} u_{m,n} - \sum_{n=m+1\to\infty} u_{m,n}$ so the 1st way limit diverges. But at this point, I think we should solidify our understanding of all this until we get to the point were Chrystal's above remarks are in fact **easy to see** and **obvious**.

Complex Double Series

From what we know above about complex series, a complex double series involves two real double series and if they converge, then the complex double series converges. If these are $\sum\alpha_{m,n}$, $\sum\beta_{m,n}$, they are abs.cnv. if $\sum\sqrt{(\alpha_{m,n}^2 + \beta_{m,n}^2)}$ is convergent. Therefore, if $u'_{m,n}$ is the modulus of $u_{m,n} = \alpha_{m,n}+ib_{m,n}$ (remember this notation for modulus) then if $\sum u'_{m,n}$ converges then $\sum u_{m,n}$ converges to the same limit in all four ways, which again Chrystal says is **obvious**. And all terms are **obviously** pos. ∴

Thm. 4.48. If all the horiz. series in $\sum u'_{m,n}$ converge and their sum of sums converge ⇒ $\sum u_{m,n}$ abs.cnv. and so are all its subsidiary series and any series constructed under RA.

Thm. 4.49. If the moduli of the series $\sum a_{m,n}x^m y^n$ have a finite upper limit λ ⇒ this ∑ is abs.cnv. for ∀x,y: x,y ∈ (0,1).

Binomial Series

We know that ∀m∈**N**

$$(1+x)^m = 1 + C_{m|1}x + C_{m|2}x^2 + \cdots + C_{m|m}x^m \qquad [1]$$

But if m∈-**N** or m∈**Q**, we no longer have m-combs of n things. If you simply follow the Binomial Theorem algorithm with these un-natural numbers, the result still holds but RHS of [1] becomes an infinite series. This will converge when:

1) x ∈ (-1,1)
2) x = 1 and m > -1
3) x = -1 and m > 0

All of this boils down to the following theorem and corollary with a **long** proof by Euler as amended by Cauchy which I leave to your curiosity and will proceed immediately here from the statement of the theorem to its use.

Thm. 4.50. Whenever the series $1 + \sum C_{m|n}x^n$ converges, its sum is the real positive value of $(1 + x)^m$.
Cor. 1. If x≠y, we can therefore always expand $(x + y)^m$ in an abs.cnv. series.

Proof

Let $x>y \therefore y/x > 1 \therefore (x + y)^m = x^m(1 + y/x)^m$

$= x^m(1 + C_{m|1}(y/x) + C_{m|2}(y/x)^2 + \cdots + C_{m|n}(y/x)^n + \cdots)$

$= x^m + + C_{m|1}x^{m-1}y + C_{m|2}x^{m-2}y^2 + \cdots + C_{m|n}x^{m-n}y^n + \cdots$ [A]

If $x < y$ and $x/y > 1 \Rightarrow (x + y)^m = y^m(1 + x/y)^m$ and Sym.

$= y^m + + C_{m|1y}{}^{m-1}x + C_{m|2}y^{m-2}x^2 + \cdots + C_{m|n}y^{m-n}x^n + \cdots$ [B]

If $m \in \mathbf{N}$ [A] and [B] both terminate and are valid. Else only one is convergent. ■

In the following examples, we assume convergence.

Examples

1) $(1 + x)^{-1} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots$

$(1 - x)^{-1} = 1 + x + x^2 + \cdots + x^n + \cdots$

$(1 + x)^{-1} = 1 + \sum C_{-1|n}x^n$

$C_{-1|n} = (-1(-1-1)(-1-2)\cdots(-1-n+1))/n!$

$= (-1)^n \cdot 1 \cdot 2 \cdots n/n! = (-1)^n \cdot 1$

$(1 - x)^{-1} = 1 + \sum C_{-1|n}(-x)^n$

$C_{-1|n}(-x)_n = (-1)^n(-1)^n x^n = x^n$

2) $(1 + x)^{-2} = 1 - 2x + 3x^2 - \cdots + (-1)^n(n+1)x^n + \cdots$

$(1 - x)^{-2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$

Since $C_{-2|n} = (-2(-2-1)\cdots(-2-n+1))/n! = (-1)^n(n+1)$

3) $(1 + x)^{-3} = 1 - 3x + 6x^2 - \cdots + (-1)^n ½(n+1)(n+2)x^n + \cdots$

$(1 - x)^{-3} = 1 + 3x + 6x^2 + \cdots + ½(n+1)(n+2)x^n + \cdots$

4) $(1 + x)^{1/2} = 1 + 1/2 \cdot x - 1/8 \cdot x^2 + \cdots + (-1)^n(1 \cdot 3 \cdot 5 \cdots (2n-3))/(2 \cdot 4 \cdot 6 \cdots 2n) \cdot x^n + \cdots$

$(1 - x)^{1/2} = 1 + 1/2 \cdot x + 1/8 \cdot x^2 + \cdots + (1 \cdot 3 \cdot 5 \cdots (2n-3))/(2 \cdot 4 \cdot 6 \cdots 2n) \cdot x^n + \cdots$

5) $(1 + x)^{-1/2} = 1 - 1/2x + 3/8x^2 - \cdots + (-1)^n(1 \cdot 3 \cdots (2n-1))/(2 \cdot 4 \cdots 2n)\, x^n + \cdots$

$(1 - x)^{-1/2} = 1 + 1/2x + 3/8x^2 + \cdots + (1 \cdot 3 \cdots (2n-1))/(2 \cdot 4 \cdots 2n)\, x^n + \cdots$

6) $(1 + x)^{m/2} = 1 + m/1 \cdot x/2 + (m(m-2))/2! \cdot (x/2)^2 + \cdots + (m(m-2)(m-4)\cdots(m-2n+2))/n! \cdot (x/2)^n + \cdots$

$= 1 + m/2 \cdot x + (m(m-2))/(2 \cdot 4) \cdot x^2 + \cdots + (m(m-2)\cdots(m-2n+2))/(2 \cdot 4 \cdots 2n) \cdot x^n + \cdots$

$(1 + x)^{-m/2} = 1 + \sum(-1)^n\, (m(m+2)(m+4)\cdots(m+2n-2))/(2 \cdot 4 \cdots 2n) \cdot x^n$

7) $(1 + x)^{p/q} = 1 + \sum(p(p-q)(p-2q)\cdots(p-nq+q))/((q2q3q\cdots nq)\, x^n$

$(1 - x)^{p/q} = 1 + \sum(p(p+q)(p+2q)\cdots(p+nq-q))/(q2q3q\cdots nq)\, x^n$

8) $(1 - x)^{-m} = 1 + \sum(m(m+1)(m+2)\cdots(m+n-1))/n!\, x^n$

9) $½((1 + x)^m + (1 - x)^m) = 1 + C_{m|2}x^2 + C_{m|4}x^4 + \cdots + C_{m|2n}x^{2n} + \cdots$

$½((1 + x)^m - (1 - x)m) = C_{m|1}x + C_{m|3}x^3 + \cdots + C_{m|2n-1}x^{2n-1} + \cdots$

And that's one of the sweetest tutorials I've ever seen. Note that in these expansions of $(1 + x)^m$ and $(1 + x)^{-m}$ are ultimately alternating. In $(1 - x)^m$, terms will have the same sign on and after term x^n where $n>m$. And in $(1 - x)^{-m}$, terms are pos. after the first term.

Consider $\sum\varphi_r(n)C_{m|n}x^n$ where φ ∀ifn of n, r°. By the process of putting one polynomial in terms of another (our change of "base" from DME) we have:

$$\varphi_r(n) = A_0 + A_1 n + A_2 n(n-1) + \cdots + A_r n(n-1)(n-r+1)$$

where A_i [0-r] are ind. of n. Then the series takes form:

$$A_0C_{m|n}x^n + mA_1xC_{m-1|n-1}x^{n-1} + m(m-1)A_2x^2C_{m-2|n-2}x^{n-2} + \cdots + m(m-1)\cdots(m-r+1)A_rx^rC_{m-r|n-r}x^{n-r}$$

And its sum is $\sum_{0\to\infty}\varphi_r(n)C_{m|n}x^n =$

$$A_0(1 + x)^m + mA_1x(1 + x)^m + \cdots + m(m - 1)\cdots(m - r + 1)A_rx^r(1 + x)^{m-r} =$$
$$(1+x)^m(A_0 + mA_1x/(1+x) + m(m-1)A_3x^2/(1+x)^2 + \cdots + m(m-1)\cdots(m-r+1)A_rx^r/(1+x)^r$$

Examples

1) Sum $m + m(m - 1)/1! + m(m - 1)(m - 2)/2! + \cdots$ if convergent

$$u_{n+1} = m(m - 1)(m - 2)\cdots(m - n)/n!$$
$$= m(m - 1)(m - 2)\cdots(m - 1 - n + 1)/n!$$
$$= mC_{m-1|n}$$

$\therefore \sum u_n = m(1 + C_{m-1|1} + C_{m-1|2} + \cdots) = m(1 + 1)^{m-1} = m2^{m-1}$ if $(m-1) > -1$ or $m > 0$

2) Evaluate $\sum_{0\to\infty}n^3C_{m|n}x^n$

$$\therefore n^3 = A_0 + A_1n + A_2n(n-1) + A_3n(n-1)(n-2)$$

```
  | 1  0   0 |  0        A₀ = 0       div 1 ⇒ 1·n = n
1 | 0  1   1
  |  1  1 |  1           A₁ = 1       div 2 ⇒ 3·n(n-1) = 3n² - 3n
2 |  0  2
     1 | 3               A₂ = 3       div 3 ⇒ 1·n(n-1)(n-2) = n³ - 3n² + 2n
                         ∴ A₃ = 1     add them to get n³
```

$\therefore \sum_{0\to\infty}n^3C_{m|n}x^n =$

$0\sum C_{m|n}x^n + 1\cdot mx\sum_{1\to\infty}C_{m-1|n-1}x^{n-1} + 3m(m-1)x^2\sum_{2\to\infty}C_{m-2|n-2}x^{n-2} + m(m-1)(m-2)x^3C_{m-3|n-3}x^{n-3}$

$= mx(1+x)^{m-1} + 3m(m-1)x^2(1+x)^{m-2} + m(m-1)(m-2)x^3(1+x)^{m-3}$

$= (m^3x^3 + m(3m - 1)x^2 + mx)(1 + x)^{m-3}$

We know that every ifrac of x can be expressed as an ifn and a proper ifrac and then every proper ifrac can be expressed as partial fractions, $\sum A(x - a)^{-n}$. Therefore any ifrac can be expressed as a series, some with ascending powers of x and some with descending.

If we denote the elementary symmetric fns $\alpha+\beta$ and $\alpha\beta$ as p,q then:

$$\alpha^n + \beta^n = a_0 p^n + a_1 p^{n-2} q + \cdots + a_r p^{n-2r} q^r + \cdots \qquad [1]$$
$$(\alpha^{n+1} - \beta^{n+1})/(\alpha-\beta) = b_0 pn + b_1 p^{n-2} q + \cdots + b_r p^{n-2r} q^r + \cdots \qquad [2]$$

and both [1],[2] terminate. We can also verify

$$\frac{2 - px}{1 - px + qx^2} = \frac{2 - (\alpha+\beta)x}{(1-\alpha x)(1-\beta x)} = \frac{1}{1 - \alpha x} + \frac{1}{1 - \beta x} \qquad [3]$$

Now take x: $px - qx^2 < 1$ and we have by the Binomial Theorem

$$[3] = (2 - px)(1 - (px - qx^2))^{-1} = (\text{letting } px-qx^2 = r)$$
$$= (2 - px)(1 + r + r^2 + \cdots + r^n + \cdots) \qquad [4]$$

Now take x: $-\alpha < x < \alpha$ where α makes $\pm px \pm qx^2 < 1$ with signs used to make this a max value. Then [4] can be expressed as a series of ascending powers of x:

$$[4] = (2 - px)(1 + \sum(p^n + C_{n-1|1} p^{n-2} q + C_{n-2|2} p^{n-4} q^2 + \cdots + (-1)^r C_{n-r|r} p^{n-2r} q^r + \cdots)x^n) \qquad [5]$$
$$= 2(1 + \sum(\text{as } [5]) - px(1 + \sum(\text{as } [5])) \qquad [6]$$

where coeff of x is

$$2(p^n - C_{n-1|1} p^{n-2} q + (-1)^r C_{n-r|r} p^{n-2r} q^r - \cdots + (-1)^r (n(n-r-1)(n-r-2)\cdots(n-2r+1)/r! \cdot p^{n-2r} q^r + \cdots)$$

and this is

$$2 + \sum(p^n - n/1! \cdot p^{n-2} q + n(n-3)/2! \cdot p^{n-4} q^2 - \cdots$$
$$+ (-1)^r (n(n-r-1)(n-r-2)\cdots(n-2r+1)/r! \cdot p^{n-2r} q^r + \cdots) \qquad [7]$$

Or from

$$1/(1-\alpha x) + 1/(1-\beta x) = (1 + \alpha x + \alpha^2 x^2 + \cdots + \alpha^n x^n + \cdots) + (1 + \beta x + \beta^2 x^2 + \cdots + \beta^n x^n + \cdots)$$
$$= 2\sum(\alpha^n + \beta^n) x^n \qquad [8]$$

which converges for x: mod αx, mod $\beta x < 1$. So by [3],[7],[8] $\therefore$

$$\alpha^n + \beta^n = p^n - n/1! \cdot p^{n-2} q + n(n-3)/2! \cdot p^{n-4} q^2 - \cdots$$
$$+ (-1)^r (n(n-r-1)(n-r-2)\cdots(n-2r+1)/r! \cdot p^{n-2r} q^r + \cdots \qquad [9]$$

And if we express $x/(1-px+qx^2)$ in terms of α,β then $(\alpha^{n+1} - \beta^{n+1})/(\alpha-\beta) =$

$$p^n - (n-1)/1!\cdot p^{n-2}q + (n-2)(n-3)/2!\cdot p^{n-4}q^2 - \cdots + (-1)^r(n-r)(n-r-1)\cdots(n-2r+1)/r!\cdot p^{n-2r}q^r + \cdots \quad [10]$$

Example

Required coeff of x^n in expansion of $(1-x)^2/(1+x)^{3/2}$

If $(1 + x)^{3/2} = 1 + \sum a_n x^n$ then $(1-x)^2/(1+x)^{3/2} = (1 - 2x + x^2)(1 + \sum a_n x^n)$
$\therefore$ coeff $= a_n - 2a_{n-1} + a_{n-2}$
If we sub actual values of these a_i this equals
$(-1)^n(16n^2 - 8n - 1)(3\cdot 5\cdots(2n - 3))/(2\cdot 4\cdots(2n))$

To go deeper into this would take us into the bowels of combinatorics, wrt C, P, and H and quickly to summing series like

$$1 - (n-3)/2! + (n-4)(n-3)/3! - (n-5)(n-6)(n-7)/4! + \cdots$$

which I leave to your curiosity. Let wisdom govern that curiosity. If combinatorics makes you happy, go for it. But any oppressive or compulsive urge to pursue something should be shaken off. Be governed by joy. If in joyfully following your interests, you discover you need to go into this (or anything else) you can always return to it with the right motive. Mathematics is an ocean you can easily drown in. Better to sport like a dolphin.

If we have $a_1x + a_2x^2 + \cdots + a_rx^r$, $x, a_i \in \mathbf{R}$ then if $\rho < 1/(a+1)$ where $a = \max a_i$ then ρ is the lower limit of the least root of $a_rx^r + \cdots + a_1x \pm 1 = 0$. Then for $\forall m \in \mathbf{R}$, if $x \in (-\rho,\rho)$

$$(1 + a_1x + \cdots + a_rx^r)^m = 1 + \sum C_{m|s}(a_1x + a_2x^2 + \cdots + a_rx^r)^s$$

Recall that $C_{m|s}$ is the number of all the s-combs of the m a_i $\therefore$

Thm. 4.51. $\forall m, x \in (-\rho,\rho)$, $(1 + a_1x + a_2x^2 + \cdots + a_rx^r)^m =$

$$1 + \sum \frac{m(m-1)\cdots(m-\sum a_i + 1)}{a_1!a_2!\cdots a_r!} a_1^{\alpha 1}a_2^{\alpha 2}\cdots a_r^{\alpha r}x^n$$

with summation over all $\alpha_i \in \mathbf{N}$: $1\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r = n$. But don't panic. Here's an

Example

Required coeff of x^n in $F = (1 + x + x^2 + \cdots + x^r)^m$

$$F = ((1 - x^{r+1})/(1 - x))^m$$
$$= (1 - x^{r+1})^m (1 - x)^{-m}$$
$$= (1 - x^{r+1})^m (1 + \sum H_{m|n} x^n)$$

$\therefore$ $n < r+1$ coeff $= H_{m|n} = m(m+1)\cdots(m+n-1)/n!$
$n \geq r+1$ coeff $= H_{m|n} - C_{m|1}H_{m|n-r-1} + C_{m|2}H_{m|n-2r-2} - \cdots$ and now you can panic (or go back and study your H's).

Exponential Series

In what follows, we are going to use a new shorthand where $\lambda \equiv \log_e a \equiv \ln a$ where e is, of course Euler's constant. If we assume a convergent expression of a^x we can determine the coeff of

$$a^x = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n + \cdots \qquad [1]$$

By our previous methods

$$L\ (a^{x+h} - a^x)/h = A_1 + 2A_2x + \cdots + nA_nx^{n-1} + \cdots$$

where RHS converges if [1] converges. Then

$$LHS = a^x\lambda L(e^{\lambda h}-1)/\lambda h = \lambda a^x$$
$$\therefore \lambda a^x = 1A_1 + 2A_2x + \cdots + nA_nx^{n-1} + \cdots \qquad [2]$$

And by [1],

$$\lambda(1A_1 + 2A_2x + \cdots + nA_nx^{n-1} + \cdots) = 1A_1 + 2A_2x + \cdots + nA_nx^{n-1} + \cdots \quad [3]$$

Both series in [3] converge $\therefore$

$$1A_1 = \lambda A_0 \qquad 2A_2 = \lambda A_1 \qquad \cdots \qquad A_n = \lambda A_{n-1}$$
$$\therefore \qquad A_1 = A_0\lambda/1! \quad A_2 = A_0\lambda^2/2! \ \cdots \quad A_n = A_0\lambda^n/n! \qquad [4]$$

Let $x = 0$ and $A_0 = 1$ [5]

$$\therefore \qquad a^x = 1 + \lambda x/1! + (\lambda x)^2/2! + \cdots + (\lambda x)^n/n! \qquad [6]$$

$\therefore$ [6] and [2] converge for $\forall x$. Let's deduce this another way using the Binomial Theorem. Let $z > 1$:

$$(1 + 1/z)^{zx} = 1 + zx\cdot 1/z + (zx(zx-1))/2!\cdot 1/z^2 + (zx(zx-1)\cdots(zx-n+1))/n!\cdot 1/z^n + \cdots$$
$$= 1 + x + (x^2(1-1/zx))/2! + \cdots + x^n(1-1/zx)\cdots(1- (n-1)/zx)/n! + R_n$$

Let x be a given quantity, n∈**Z**, then z can be taken as large as possible such that zx∈**N**, ρ>n then R_n terminates. ∴

$$R_n < x^{n+1}/(n+1)! \cdot (1 - (x/(n+2))) \quad [3]$$

$$\therefore (1 + 1/z)^{zx} = 1 + x + x^2(1-1/\rho)/2! + \cdots + x^n(1-1/\rho)\cdots(1 - (n-1)/\rho)/n! + R_n \quad [4]$$

$z\to\infty \Rightarrow \rho\to\infty$ $L_{z\to\infty}(1 + 1/z)^{zx} = 1 + x + x^2/2! + \cdots + x^n/n! + R_n = e^x$ [5]
$n\to\infty \Rightarrow [3]\to 0$ ∴ [5] without remainder ≡ $L_\infty(1 + 1/z)^{zx}$
Note that [5] is used to give an upper limit to the residue of the series. We could use this to calculate any approximate value of e. Going to n=12, e = 2.718281829.

Now let's show e is incommensurable, irrational.
Proof
Else $e = p/q \in$ **Q** and $e = L_\infty(1 + 1/z)^z$
∴ $p/q = 2 + 1/2! + \cdots + 1/q! + R_q$
∴ $R_q < (q+2)/(q+1)^2 q!$
$q!R_q < (q+2)/(q+1)^2 < (q+2)/(q(q+2) + 1)$
∴ $q!R_q$ is a positive proper fraction ↴
∴ e is incommensurable, irrational, ... ■

Let's look at e the way Cauchy did. We will take the sum of its infinite series f(x), x∈**C**

$$f(x) = 1 + x + x^2/2! + \cdots + x^n/n! + \cdots$$

This converges ∀x ∴ f(x) is a single-valued, finite, continuous fn of x
∀y∈**C** we have f(x)×f(y) =

$$1 + (x+y) + (x^2/2!+ xy/1!1! + y^2/2!) + \cdots + (x^n/n! + x^{n-1}y/(n-1)! + \cdots + y^n/n!) + \cdots$$

You should recall this next bit, in more general terms, from DME:

This has an nth term = $(x^n+ C_{n|1}x^{n-1}y + C_{n|2}x^{n-2}y^2 + \cdots + y^n)/n!$
∴ $f(x)\times f(y) = 1 + \sum(x+y)^n/n! = f(x+y)$ [1]
$f(x)f(y)f(z) = f(x+y)f(z) = f(x+y+z)$ ∴

Thm. 4.52. Addition Theorem for Exponential Series
$f(x)f(y)f(z)\cdots = f(x + y + z + \cdots)$ [2]
Proof
Let each of these vars equal unity and let there be n of them.
∴ $(f(1))^n = f(n)$ [3]
Or let each of the letters be p/q with q of them, p,q∈**N**
∴ $(f(p/q))^n = f(p)$ [4]
∴ $(f(p/q))^q = (f(1))^p$ (by [4],[3]) [5]
In [1], let y = -x
∴ $f(x)f(-x) = f(0)$ [6]
From [5],[6], we can sum the series for ∀x∈**Q**, In [5], f(p/q) is the qth root of $(f(1))^p$.

But $p/q > 0$, ∴ $f(p/q) \in \mathbf{R}^+$ and

$$f(1) = 1 + 1/1! + 1/2! + \cdots = e \in \mathbf{R}^+$$

∴ $f(p/q)$ is the real positive root of e^p and equals $e^{p/q}$:

$$1 + (p/q)/1! + (p/q)^2/2! + \cdots = e^{p/q} \qquad [7]$$

$f(0) = 1$. So from [6], $f(-p/q) = 1/f(p/q) = 1/e^{p/q} = e^{-p/q}$ ∴

$$e^{-p/q} = 1 + (-p/q)/1! + (-p/q)^2/2! + \cdots \qquad [8]$$

Combining [7],[8]:

$$e^x = 1 + x/1! + x^2/2! + \cdots + x^n/n! + \cdots$$
$$e = 1 + 1/1! + 1/2! + \cdots + 1/n! + \cdots$$

And that's all I have to say about that. ■

Bernoulli's Numbers

Thm. 4.53. Cauchy's Expansion of $x/(1 - e^{-x})$
Proof and Statement Combined

$x/(1 - e^{-x}) = 1/((1 - e^{-x})/x) = 1/(1 - y)$ [1]
where $y = 1/((1 - e^{-x})/x)$ [2]
∴ $x/(1 - e^{-x}) = 1 + y + y^2 + \cdots + y^n + \cdots$ [3]
which is convergent for $y \in (-1,1)$ and from [2] and the Exponential Thm.
∴ $y = x/2! - x^2/3! + x^3/4! - \cdots$ which is abs.cnv. for ∀x
So we need ρ: $\rho/2! + \rho^2/3! + \rho^3/4! + \cdots < 1$ [A]
so that if we sub [4] into [3] it is abs.cnv.
But [A] = $(e^\rho - 1)/(\rho - 1)$. So we need ρ: $e^\rho - 1 < 2\rho$ [5]
From the graphs of $e^x - 1$ and $2x$, [5] is true if ρ < unique pos. root of $e^x - 1 = 2x$
which is at (1,2) ∴ $x/(1 - e^{-x})$ series is convergent for $x \in (-1,1)$. So sub y into [3]:
∴ $x/(1-e^{-x}) = 1 + ½x + 1/6 \cdot x^2/2! - 1/30 \cdot x^3/4! + 1/42 \cdot x^4/6! - \cdots$ [6] ■

If we use our method of determining coeffs A_i of equal series (which is quite involved here) we arrive at a formula for calculating these coeffs (1/2, 1/6, ...). And what we end up with are **Bernoulli's Numbers** B_i [1-n] where $x/(1 - e^{-x})$ =

$$1 + ½x + B_1x^2/2! - B_2x^4/4! + B_3x^6/6! - \cdots$$

You're going to love this. For odd n we calculate B_n with

$$C_{2n+1|2n}B_n - C_{2n+1|2n-2}B_{n-1} + \cdots + (-1)^{n-1}C_{2n+1|2}B_1 = (-1)^{n-1}(n - ½)$$

and for even n

$$C_{2n+2|2n}B_n - C_{2n+2|2n-2}B_{n-1} + \cdots + (-1)^{n-1}C_{2n+2|2}B_1 = (-1)^{n-1}n$$

Cor. 1. From $(x(e^x + e^{-x}))/(e^x - e^{-x}) = x/(1 - e^{-2x}) - x/(1 - e^{2x})$ we get

$$LHS = 1 + B_1/2!\cdot2^2x^2 - B_2/4!\cdot2^4x^4 + B_3/6!\cdot2^6x^6 - \cdots$$

Cor. 2. From $x/(1 + e^{-x}) = 2x/(1 - e^{-2x}) - x/(1 - e^{-x})$ we get

$$LHS = ½(2^1 - 1)x + B_1/2!\cdot(2^2 - 1)x^2 - B_2/4!\cdot(2^4 - 1)x^4 + \cdots$$

Back in combinatorics, we had the sum of the first n rth powers $S_{n|r}$. Using the series of $(e^{nx} - 1)/(1 - e^{-x})$ we have:

Thm. 4.54. Bernoulli's Theorem
$S_{n|r} = n^{r+1}/(r+1) + ½n^r + r/2!\cdot B_1n^{r-1} - r(r-1)(r-2)/4!\cdot B_2n^{r-3} + r(r-1)\cdots(r-4)/6!\cdot B_3n^{r-5} - \cdots$
where last term even or odd is either $(-1)^{½(r-2)}B_{½r}n$ or $½(-1)^{½(r-3)}rB_{½(r-1)}n^2$

If we pursued this further, we would find

Thm. 4.55. We can always sum the infinite series

$$\sum\varphi_r(n)x^n/n!$$

where $\varphi_r(n)$ is an ifn of n r°
Cor. 1. We can generally sum

$$\sum\varphi_r(n)x^n/n!\ (n+a)(n+b)\cdots(n+k)$$

where a-k unequal $\in$**N** and this would allow us to do things like sum

$1^3/1!\cdot x + (1^3 + 2^3)/2!\cdot x^2 + \cdots + (1^3 + 2^3 + \cdots + n^3)/n!\cdot x^n + \cdots$ which comes to 27e/4.

Logarithmic Series

Consider the expansion of ln(1 + x). No f(x) where f(1) = ∞ can be expanded into a convergent series of ascending powers of x. But

Thm. 4.56. We can expand ln(1 + x) if x<1 in ascending powers of x.

[proof follows]

Proof

$(1 + x)^z = 1 + z(\ln(1+x)) + z^2(\ln(1+x))^2/2! + \cdots$ [1]

which is convergent ∀z. And if x<1, by the Binomial Theorem

$(1 + x)^z = 1 + zx - z(1 - z/1)x^2/2 + z(1 - z/1)(1 - z/2)x^3/3 + \cdots$

Recall our earlier $P_{n|m}$. As a double series, that last bit is

$$(1 + x)^z = 1 + zx - (zx^2/2 - z^2x^2/2) - \cdots$$
$$+ (-1)^{n-1}(zx^n/n - P_{n-1|1}z^2x^n/n + P_{n-1|2}z^3x^n/n - \cdots + (-1)^{n-1}P_{n-1|n-1}n^nx^n/n)$$

where $P_{n-1|r}$ is sum of all r-products of 1/1, 1/2, ..., 1/(n-1) without repetition, in case you forgot. By Cauchy's abs.cnv. theorem, the above will converge when z,x > 0. The series with the $P_{n|m}$ sums to $z(z+1)\cdots(z+n-1)x^n/n$ and goes to 0 when n→∞ and the 1 + zx series before it converges when x<1.

$$\therefore\ (1 + x)^z = 1\ (\ (x/1 - x^2/2 + x^3/3 - \cdots)z + \cdots$$

Because an expansion in powers of z must be unique, this is equivalent to [1]

$$\therefore\ \ln(1 + x) = x/1 - x^2/2 + x^3/3 - \cdots (-1)^{n-1}x^n/n$$

which is the logarithmic series convergent on (-1,1) ■

Cor. 1. $(\ln(1+x))^n = n!(P_{n-1|n-1}x^n/n - P_{n|n-1}x^{n+1}/(n+1) + P_{n+1|n-1}x^{n+2}/(n+2) - \cdots$

Cor. 2. $\ln(1-x) = -x/1 - x^2/2 - x^3/3 - \cdots - x^n/n - \cdots$

Cor. 3. $\ln(\ (1+x)/(1-x)\ ) = 2(x/1 + x^3/3 + \cdots + x^{2n-1}/(2n-1)$ by subtraction of logs.

Thm. 4.57. The series with nth term $\varphi(n)x^n/(n+a)(n+b)\cdots(n+k)$ where φn ifn of n and a-k unequal ∈**Z** can be summed if series converges.Which would allow us to sum $\sum_{2\to\infty}n^3x^n/(n+1)(n+2)$. We already know, if x > 0,

$$x - 1 < \ln x < 1 - 1/x \qquad [1]$$

Example

Show ln n/(m-1) > 1/m + 1/(m+1) + 1/(m+2) + ⋯ + 1/n > ln (n+1)/m

If we put 1 - 1/x = 1/m ⇒ x = m/(m-1) in middle term of [1].

Then replace m successively with m+1, m+2, ... and we get

$$\ln m - \ln(m-1) > 1/m$$
$$\ln(m+1) - \ln m > 1/(m+1)$$
$$\cdots$$
$$\ln n - \ln(n-1) > 1/n \qquad [2]$$

By addition, ln n - ln(m-1) > 1/m + 1/(m+1) + 1/(m+2) + ⋯ + 1/n

Then in LHT [1] put x - 1 = 1/m and Sym.

$$\ln(n+1) - \ln m < 1/m + 1/(m+1) + 1/(m+2) + \cdots + 1/n \qquad [3]$$

and result follows from [2],[3]

Recurring Series

The next three chapters in Chrystal deal almost exclusively with complex-valued series representing trigonometric forms of number. You have already seen the core of this in the hyperbolic trig chapter of DME. I will save all these trig series for the Second Circle of Trigonometry where these ideas will lead us into Complex Analysis. But here at the end of these chapters are a couple of useful ideas apart from trig and because they include ifracs, I **have** to share them with you. Ifracs as **soooo** cool.

∀ifrac, such as

$$(a + bx + cx^2)/(1 + px + qx^2 + rx^3) \qquad [1]$$

can **always** be expanded, as we know, in ascending powers of x. If mod x < root with the least modulus for the denom then the series for [1] has form:

$$u_0 + u_1x + u_2x^2 + \cdots + u_nx^n + \cdots \qquad [2]$$

If we set [1] = [2] and multiply both side by the denom of [1], we get

$$a + bx + cx^2 = (u_0 + u_1x + u_2x^2 + \cdots + u_nx^n + \cdots)(1 + px + qx^2 + rx^3)$$

and be equating powers of x

$$u_0 = a$$
$$u_1 + pu_0 = b$$
$$u_2 + pu_1 + qu_0 = c$$
$$u_3 + pu_2 + qu_1 + ru_0 = 0$$
$$\cdots$$
$$u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0 \qquad [3]$$

Any power series where form [3] arises is a **recurring power series** (rec series) and the explicit form that [3] takes is the series' **scale** and will have 1-r constants (here, three of them: p,q,r) and the number of constants is the series' **order**. All the coeffs of an rth order rec series can be derived from the r coeffs in its scale ([3]) A rec series of rth order depends upon 2r constants: r in the scale and r more. So if the first 2r terms are given, the series can be continued as a rec series of rth order in only one way. OR as an (r+1)th order rec series this series becomes a two-fold infinity. OR two conditions must be satisfied to continue our series as an (r-1)th order rec series, 4 conditions for (r-2)th order and so on.

Example

Show that $x + 2x^2 + 3x^3 + 4x^4 + \cdots$ has order 2. Let the scale be $u_n + pu_{n-1} + qu_{n-2}$ ∴

$$3 + 2p + q = 0 \qquad 5 + 4p + 3q = 0$$
$$4 + 3p + 2q = 0 \qquad 6 + 5p + 4q = 0$$

Soln of the first two is p = -2, q = 1 and these values solve the others

Now think about [2] as being generated by [1]. Then [1] is the **generating fn** (gen fn) of the series. The denom furnishes the scale and the coeffs are determined by the r eqns leading up to [3]. So:

Thm. 4.58. Given the scale and the first r terms of a rec series we can determine the gen fn of the series.
Proof
Pick an r, any r, say 3. From above $(u_0+(u_1+p_0)x+(u_2+pu_1+qu_0)x^2$/denom is the gen fn of [2] with scale [3]. ■
Cor. 1. Every recurring power series, if mod x taken properly small, is the expansion of an ifrac.
Cor. 2. The general term of a rec series can be found given the scale and r terms.

In other words, to find the gen fn of a rec series **or** a corresponding power series, decompose the gen fn into partial fractions (yay!) in the form of $A(x - a)^{-s}$, expand these into ascending powers of x, and collect the coeff of x^n for the general term.

Example

Given scale $u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3}$ and first three terms 1, + 0, - 5, find general term.
$p = -4 \quad q = 5 \quad r = -2$
$a = u_0 = 1 \quad b = u_1 + pu_0 = -4 \quad c = u_2 + pu_1 + qu_0 = 0$
$\therefore$ gen fn $\equiv (1 - 4x)/(1 - 4x + 5x^2 - 2x^3)$
$= (1 - 4x)/(1 - x)^2(1 - 2x)$
$= 2/(1-x) + 3/(1-x)^2 - 4/(1-2x)$
$= 2(1 + \sum x^n) + 3(1 + \sum(n+1)x^n) - 4(1 + \sum 2^n x^n)$
$= 1 + \sum(3n + 5 - 2^{n+2})x^n$
$\therefore$ general term $\equiv 3n + 5 - 2^{n+2}$

If u_n is a fn of n satisfying

$$u_n + pu_{n-1} + pu_{n-2} + ru_{n-3}$$

which is also

$$u_{n+3} + pu_{n+2} + qu_{n+1} + ru_n \qquad [1]$$

then given u_0, u_1, u_2, [1] uniquely determines the fn u_n but here any other three u_i would suffice. So we determine u_n as $f(u_0,u_1,u_2,n)$ where u_0, u_1, u_2 determined by

$$f(u_0,u_1,u_2,\alpha) = u_\alpha$$
$$f(u_0,u_1,u_2,\beta) = u_\beta$$
$$f(u_0,u_1,u_2,\gamma) = u_\gamma$$

So our [1] is a **linear difference eqn** order 3 which has a unique soln given three values of its linear argument. And so on for rth order...

Example

Required u_n where the scale is $u_{n+3} - 4u_{n+2} + 5u_{n+1} - 2u_n$ and $u_0, u_1, u_2 = 1, 0, -5$
See last example.

We can sum a rec series to n+1 terms for finite n or to hell and gone if convergent. Take on of order 3.

$$S_n = u_0 + u_1x + u_2x^2 + u_3x^3 + \cdots + u_nx^n$$

∴

$$pxS_n = pu_0x + pu_1x^2 + \cdots + pu_{n-1}x^n + pu_nx^{n+1}$$
$$qx^2S_n = \quad qu_0x^2 + qu_1x^3 + \cdots + pu_{n-2}x^n + pu_{n-1}x^{n+1} + pu_nx^{n+2}$$
$$rx^3S_n = \quad ru_0x^3 + \cdots + ru_{n-3}x^n + ru_{n-2}x^{n+1} + ru_{n-1}x^{n+2} + ru_nx^{n+3}$$

Summing and recalling when scale equals zero we get

$$(\text{denom})\cdot S_n = u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2 + (pu_n + qu_{n-1} + ru_{n-2})x^{n+1} + (qu_n + ru_{n-1})x^{n+2} + ru_nx^{n+3}$$

Then divide by the denom and we have the sum. Consider the case where x = a is a root of the denom. Then $S_n = 0/0$ and we can, if we must, evaluate this by our earlier methods for handling 0/0. Now if our above series converges then $Lu_0x^n = 0$ and the last three terms just above go to zero.

$$S_\infty = (u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2)/(1 + px + qx^2 + rx^3)$$

We have already dealt with such series:

$\sum n^2x^n$: scale $\equiv u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$
$1 + \sum(-1)^{n-1}2nx^n$: scale $\equiv u_n + 2u_{n-1} + u_{n-2} = 0$

We end our prolonged exploration of series with **Euler's** (obvious) **Identity**:

$$1 - a_1(1-a_2) + a_1a_2(1-a_3) + a_1a_2\cdots a_n(1 - a_{n+1}) = 1 - a_1a_2a_3\cdots a_{n+1} \qquad [1]$$

which, amazingly, is kind of obvious. So I must be getting smarter. If in [1] we sub:

$$a_1 = x/y \quad a_2 = (x + p_1)/(y + p_1) \quad \cdots \quad a_{n+1} = (x + p_n)/(y + p_n)$$

$$\therefore\ 1 + x/(y+p_1) + x(x+p_1)/(y+p_1)(y+p_2) + \cdots + x(x+p_1)\cdots(x+p_{n-1})/(y+p_1)\cdots(y+p_n)$$
$$= y/y\text{-}x - x/y\text{-}x \cdot ((x+p_1)(x+p_2)\cdots(x+p_n))/((y+p_1)\cdots(y+p_n)) \qquad [2]$$

Let that final fraction factor be F, then $L_{n\to\infty}F = 0$ and then LHS, taken to ∞, becomes y/(y - x), and if in [2], y = 0:

$$L_\infty\ 1 - x/p_1 + x(x-p_1)/p_1p_2 - \cdots = 0$$

Thm. 4.59. If $\prod_{1\to\infty} (1 + x/p_n)$ converges to a finite limit then

$$1 + \sum_{1\to\infty}(x+p_1)(x+p_2)\cdots(x + p_{n-1})/p_1p_2\cdots p_n$$

converges to the same limit.

Example

If $S = 1 + x/y+p + x(x+p)/(y+p)(y+2p) + x(x+p)(x+2p)/(y+p)(y+2p)(y+3p) + \cdots$
when does S converge and to what limit?

$$S = y/y\text{-}x - x/y\text{-}x \cdot L_\infty\ (x+p)(x+2p)\cdots(x+np)/(y+p)(y+2p)\cdots(y+np)$$

And this limit is $\prod_{1\to\infty}(1 + ((x\text{-}y)/np)/(1 + y/np)$ which diverges if $(x\text{-}y)/np > 0$ and converges to zero if $(x\text{-}y)/np < 0$. So whether y >< x, this sums to $y/y\text{-}x$.

5. Continued Fractions

I have a thing for continued fractions (CF). They are another example of everything interesting arising from division. And they are just full of the GCM. So while we have already met them in DME, I'm allowing myself the pleasure of taking them from the top.

Simple Continued Fractions

In Chrystal, an everyday CF has the form:

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}} \qquad [1]$$

where b_i is the antecedent and its consequent is everything below it. This consequent is either **finite** ≡ **terminating** (term) or **infinite** ≡ **non-terminating** (nonterm). The component fractions b_2/a_2, b_3/a_3, ... can have pos. or neg. nums and denoms and need express no law of recurrence. If they do fall under such a law, they are **periodic** ≡ **recurring** (RCF). Sometime, for clarities sake, we will use this notation:

$$a_1 + \frac{b_2}{a_2 +}\ \frac{b_3}{a_3 +}\ \ldots$$

with the + signs below to avoid confusion. We know from DME that if $\forall a_i, b_i \in \mathbf{Q}$ and the CF terminates then the CF reduces to some $q \in \mathbf{Q}$. We will begin with simple CF (SCF) where $\forall b_i = 1$. This allows an even simpler notation:

$$a_1 \colon a_2, a_3, \ldots$$

An SCF can represent either a proper or improper fraction. So a_1 is the integral part of an improper fraction and is zero if proper.

Thm. 5.1. $\forall r \in \mathbf{R}$ expandable as an SCF which may or may not terminate.
Proof
(Let me point out that this proof is our algorithm or method of creating CF and will reappear as it is expanded to work for non-simple CF.)
Let X be the $r \in \mathbf{R}$, a_1 max $n \in \mathbf{N} < X \Rightarrow$

$$X = a_1 + 1/X_1 \qquad [1]$$

where $X_1 > 1 \in \mathbf{R}$. Let a_2 be max $n \in \mathbf{N} < X_1 \Rightarrow$

$$X_1 = a_2 + 1/X_2 \qquad [2]$$

Sym. $$X_2 = a_3 + 1/X_3 \qquad [3]$$

[proof cont'd]

Now if some $X_{n-1} \in \mathbf{Z}$, this all terminates with

$$X_{n-1} = a_n$$

and our CF is a_1: a_2, a_3, ..., a_n. But if $\forall X_i \notin \mathbf{N}$ then our CF is non-terminating. Now suppose there are two equal CF

$$a_1: a_2, a_3, \ldots = a_1': a_2', a_3', \ldots = X$$

$\forall a_i, a_i'$ [2-n] where $n \in [2-\infty)$ are positive integers. So any sum of any consecutive sequence of them is a proper fraction
$\therefore$ $a_1 = a_1'$ and for any sequence $1/a_m + \cdots + 1/a_n = 1/a_m' + \cdots + 1/a_n'$
Else we would need an improper fraction arising somewhere to fix any inequality in the sequences and all our fracs are proper. So all our sums are proper fracs. ■

Cor. 1. It follows that if equality is maintained by adding $x_{n+1}, y_{n+1} \in \mathbf{R}^+$ to our two sequences, then we must have $x_{n+1} = y_{n+1}$

We saw in DME that the proposition "$\forall q \in \mathbf{Q} = A/B$ is a terminating CF" is equivalent to "$\forall q \in \mathbf{Q} = A/B$ is subject to Euclid's Algorithm."

Example

167/81 Take the GCM:

```
81 ) 167 ( 2
     162
       5 ) 81 ( 16
           80
            1) 5 ( 5
               5
               0
```

$\therefore$ 167/81 = 2: 16, 5

And 81/167 = 0: 2, 16, 5
Which shows you how to place the elements of a proper fraction into Euclid 7.1.

Any simple surd $(A + B\rho^{1/n} + C\rho^{2/n} + \cdots + K\rho^{(n-1)/n})/n$ is expandable as a nontermCF.

Examples

1) $\sqrt{13} = 3 + \sqrt{13} - 3$ where a_1, as above, = 3
$= 3 + 1/(1/(\sqrt{13} - 3)) = 3 + 1/((\sqrt{13} + 3)/4)$

Then $a_2 = 1$ from
$(\sqrt{13} + 3)/4 = 1 + (\sqrt{13} - 1)/4 = 1 + 1/(4/(\sqrt{13} - 1)) = 1 + 1/((\sqrt{13} + 1)/3)$

And so on where $\sqrt{13}$ = 3:1111611116... which we will denote 3:1,1,1,1,6,...$_R$

2) $(\sqrt{3} - 1)/2 = 0 + 1/(2/(\sqrt{3} - 1)) = 0 + 1/(\sqrt{3} + 1)$
$\sqrt{3} + 1 = 2 + \sqrt{3} - 1 = 2 + 1/(1/(\sqrt{3} - 1)) = 2 + 1/(\sqrt{3} + 1)/2)$
and if you do the next step you can see that it will begin to repeat
$\therefore (\sqrt{3} - 1)/2 = 0: 2, 1, \ldots_R$

What we are showing is that every number in **R** can take the form of a simple CF. It is only a convention that we generally consider the form 167/81 or its electro-mechanical equivalent 2.06173 rather than 2:16, 5. My calculator can show 167/81 as a 2 followed by thirteen digits. Does it terminate there? But our 2: 16, 5 **terminates** with the 5.

For any i we can define x_i =

$$a_i + \frac{1}{a_{i+1} +}\ \frac{1}{a_{i+2} + \cdots}$$

as the ith **complete quotient**. where a_i is the integral part of x_i. Mostly we consider the **convergents** p_i/q_i :

$$\begin{aligned} a_1 &= a_1/1 && && = p_1/q_1 \\ a_1{:}a_2 &= a_1 + 1/a_2 && = (a_1a_2 + 1)/a_2 && = p_2/q_2 \\ a_1{:}a_2,a_3 &= (a_1a_2a_3 + a_1 + a_3)/(a_2a_3 + 1) && && = p_3/q_3 \end{aligned}$$

and so on. Note that

$$\begin{aligned} p_1 &= a_1 & q_1 &= 1 \\ p_2 &= a_1a_2+1 & q_2 &= a_2 \\ p_3 &= a_1a_2a_3+a_1+a_3 & q_3 &= a_2a_3+1 \end{aligned}$$

and so on and these p_i/q_i are the ith convergents. And if the CF terminates, the last convergent is the CF itself. So as soon as we have two convergents, we can derive the rest:

$$p_n = a_np_{n-1} + p_{n-2} \qquad [1]$$
$$q_n = a_nq_{n-1} + q_{n-2} \qquad [2]$$

And you can prove this by a simple induction calculation. Because $a_n \in \mathbf{N}$, we can see that both the num and denom of our convergents are monotonically increasing sequences of integers. And by division here and subbing n-1, n-2, ... for n we get:

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} +}\ \frac{1}{a_{n-2} +} \cdots \frac{1}{+\ a_1}$$

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1} +}\ \frac{1}{a_{n-2} +} \cdots \frac{1}{+\ a_2}$$

Examples

1) The convergents of π are 3/1, 22/7, 333/106, 355/113, 103993/33102, ...

2) Given $a_1: a_2, a_1, a_2, a_1, \ldots_R$
Show $p_{2n} = q_{2n+1}$ and $p_{2n-1} = a_1 q_{2n}/a_2$

By def. (just above, with + signs)

$$p_{2n+1}/q_{2n+1} = a_1{:}a_2, \ldots, a_1$$

because every odd convergent will end in $1/a_1$. Also,

$$p_{2n+1}/p_{2n} = a_1{:}\ a_2, \ldots, a_1$$

$\therefore$ $$p_{2n+1}/q_{2n+1} = p_{2n+1}/p_{2n} \ \therefore\ q_{2n+1} = p_{2n} \qquad [3]$$

Then from $p_{2n} = a_2 p_{2n-1} + p_{2n-2}$
$q_{2n+1} = a_1 q_{2n} + q_{2n-1}$

We get $$a_2 p_{2n-1} + p_{2n-2} = a_1 q_{2n} + q_{2n-1} \qquad [4]$$

In [3], sub n-1 for n $\therefore$ [4] gives us $a_2 p_{2n-1} = a_1 q_{2n}$ and second result follows.

Thm. 5.2. From [1],[2] above we can derive: $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ (and this is an **important** result).
Cor. 1. Convergents derived by the method of Thm. 5.1. are fractions in lowest terms.
Proof
If p_n, q_n have a common factor it must divide $(-1)^n$ exactly $\therefore$ $p(p_n,q_n)$ $\therefore$ lowest terms. ■
Cor. 2. $p_n/q_n - p_{n-1}/q_{n-1} = (-1)^n/q_n q_{n-1}$
Cor. 3. $\dfrac{p_n}{q_n} = a_1 + \dfrac{1}{q_1 q_2} - \dfrac{1}{q_2 q_3} + \cdots + \dfrac{(-1)^n}{q_{n-1} q_n}$
Cor. 4. $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$
Cor. 5. $p_n/q_n - p_{n-2}/q_{n-2} = (-1)^{n-1} a_n/q_n q_{n-2}$
Cor. 6. The odd convergents monotonically increase. Even convergents monotonically decrease. ∀even convergent > ∀odd convergent. ∀odd/even convergent </> next convergent.
Cor. 7. $\forall p,q \in \mathbf{Z}$: $p(p,q)$ $\exists p',q' \in \mathbf{N}$: $pq' - p'q = \pm 1$
Proof
Let p'/q' be next to last convergent (last one is our $p/q \in \mathbf{Q}$) then $pq' - p'q = 1$ if partial quotients are even and -1 if odd. And by letting the numerators not equal unity (i.e. the CF need not be an SCF), we can have even or odd by choice. ■

Thm. 5.3. Convergents of odd order are each less than the CF and those of even order are each greater than the CF. Each convergent is closer in value to the CF than any preceding convergent.
Cor. 1. Let the difference between the nth convergent and the CF be $\delta \Rightarrow$

$$a_{n+2}/q_n q_{n+1} < \delta < 1/q_n q_{n+1}$$

Cor. 2. The odd/even convergents form an increasing/decreasing series of rational fractions approaching the value of the CF.

Cor. 3. Recall x_1 as the complete quotient. If $p_n/q_n - x_1 < 1/(q_n(q_n+q_{n-1}))$ where n is the last index and n-1 penultimate and the SCF has an even number of convergents ⇒ p_n/q_n is a convergent of the SCF of x_1. And if $x_1 - p_n/q_n < 1/(q_n(q_n+q_{n-1}))$ where CF has an odd number of convergents the same holds.

It follows that a CF gives both a method of continually improving exactness in approximation and a method of estimating that exactness at any point. Neither can be said of positional decimal notation.

Chrystal points out that, arithmetically, we could calculate termCF in reverse. *In the case of non-terminating CF, no such alternative course is, strictly speaking, open to us. Indeed, the further difficulty arises that, a priori, we have no certainty that such a continued fraction has any definite meaning at all.* But the nontermCF ∈ **R-Q** is the expression of a law. It is the value of the limit of its two series. Or rather, one series has for its last term the limit and the other approaches it ad infinitum. And you have the tools to work out this law for yourself. Every simple continued fraction has a definite finite limit.

Problem
To find the fraction whose denom by absolute value does not exceed some D ∈ **N** which shall most closely approximate, by excess or defect as designated, a given value.
Method
Lemma p/q, p'/q' ∈ **Q**: pq' -p'q = 1 ⇒ no fraction p"/q" can lie between them unless q" > max(q,q')
Proof
a/b: p/q > a/b > p'/q'
∴ p/q - a/b < p/q - p'/q' [1]
a/b - p'/q' < p/q - p'/q' [2]
From [1] (pb - qa)/qb < (pq' - p'q)/qq' ∴ (pb-qa)/qb < 1/qq'
∴ qb > qq'(pb - qa)
∴ b > (pb - qa)q'
p/q - a/b > 0 ∴ pb - qa > 0 ∈ **N** ∴ b > q'
Sym. from [2] b > q ■

Given any two convergents, we call the one with the larger denom **more complex**. It follows from above that the nth convergent is a nearer approximation to its CF than any fraction with a smaller or equal denom. Let x_1 be the final value and consider these convergents:

$$\frac{p_{n-2}}{q_{n-2}} \quad \frac{p_n}{q_n} \quad x_1 \quad \frac{p_{n-1}}{q_{n-1}}$$ and call this a c x_1 b

We have shown that a such a fraction without a greater denom cannot exist on (a,b) and (b,c). But what about (a,c)?

Thm. 5.4. The series of fractions

$$\frac{p_{n-2}}{q_{n-2}} + \frac{p_{n-2} + p_{n-1}}{q_{n-2} + q_{n-1}} + \frac{p_{n-2} + 2p_{n-1}}{q_{n-2} + 2q_{n-1}} + \cdots + \frac{p_{n-2} + (a_n - 1)p_{n-1}}{q_{n-2} + (a_n - 1)q_{n-1}} + \frac{p_n}{q_n} \text{ (which equals } \frac{p_{n-2} + a_n p_{n-1}}{q_{n-2} + a_n q_{n-1}}\text{)}$$

forms a monotonically increasing/decreasing series as n odd/even. Each element of the series is in lowest terms. Each consecutive pair (P/Q,P'/Q') satisfies $PQ' - P'Q = \pm 1$ so that no $q \in \mathbf{Q}$ less complex than the most complex of the pair can lie between them.

Proof

To prove this to yourself, show

$P/Q - P'/Q' = (p_{n-2} + rp_{n-1})/(q_{n-2} + rq_{n-1}) - (p_{n-2} + (r+1)p_{n-1})/(q_{n-2} + (r+1)q_{n-1}) = \pm 1/QQ'$

If we call p_{n-2}/q_{n-2} and p_n/q_n here the **principal convergents** then the others are **intermediate convergents**. At this point, a friendly suggestion. Take an SCF, calculate its convergents, then take two principal convergents and calculate their intermediates. This will prevent mental muddles.

Still with regard to our **Problem**, we can further show that since

$$P/Q - p_{n-1}/q_{n-1} = \pm q_{n-1}(q_{n-2} + rq_{n-1})$$
$$P/Q <> x_1 <> p_{n-1}/q_{n-1}$$

then the intermediate convergent P/Q differs from the CF by less than $1/q_{n-1}Q$ so by less than $1/q_{n-1}$. The point here is that we have two series:

$$0/1,\ p_1/q_1,\ p_3/q_3,\ \ldots,\ p_n/q_n \qquad [1]$$
$$1/0,\ p_2/q_2,\ p_4/q_4,\ \ldots,\ p_{n-1}/q_{n-1} \qquad [2]$$

[1] monotonically increases and [2] monotonically decreases and you can't insert any q in **Q** closer to x_1 that is less complex than the more complex of any two consecutive elements of either series. So our **Problem**'s soln is this:

If we want to approximate by defect, take p_i/q_i in [1] where q_i is the max denom in the series less than D, and if by excess then Sym. take this element from [2]. In the first case q_i may equal D. If the denom in [1] which is the max $q_i < D$ is an intermediate convergent (I told you to do an example) then the corresponding q_i in [2] in the denom of a principal convergent. ■

Recurring Continued Fractions

Every simple quadratic surd is a recurring continued fraction (RCF). Consider any such surd. We can always do the following to it and these results are generally true. We consider:

$$\tfrac{1}{4}(2 - \sqrt{(3/2)}) = \tfrac{1}{4}(2 - \sqrt{6}/2) = -4 + \sqrt{6}/-8 = (-16 + \sqrt{96})/-32$$

And thus we arrive at our standard form of $(P_1 - \sqrt{R})/Q_1$ where $P_1 = -16$, $Q_1 = -32$, and $R = 96$. And in this form, it will always hold that $R - P_1^2$ divby Q_1, here $96 - 256 = -160$

which is divby -32. Of course, there are proofs of these assertions and you know where to find them. Given a surd brought to this state our same algorithm takes this form:

$$x_1 = (P_1 + \sqrt{R})/Q_1 = a_1 + 1/x_2$$
$$x_2 = (P_2 + \sqrt{R})/Q_2 = a_2 + 1/x_3$$
$$\ldots$$
$$x_n = (P_n + \sqrt{R})/Q_n = a_n + 1/x_{n+1}$$

where a_i is max $n \in \mathbf{N}$ etc. as before. We have

$(P_n + \sqrt{R})/Q_n = a_n + 1/((P_{n+1} + \sqrt{R})/Q_{n+1})$
$\therefore ((P_n - a_nQ_n)P_{n+1} - Q_nQ_{n+1} + R) + (P_n - a_nQ_n + P_{n+1})\sqrt{R} = 0$
$\therefore (P_n - a_nQ_n)P_{n+1} - Q_nQ_{n+1} + R = 0 \wedge P_n - a_nQ_n + P_{n+1} = 0$
$\therefore P_{n+1} = a_nQ_n - P_n$ [A]
$P_{n+1}^2 + Q_nQ_{n+1} = R \equiv P_n^2 + Q_{n-1}Q_n = R$
$\therefore Q_nQ_{n+1} = R - (a_nQ_n - P_n)^2 = P_n^2 + Q_{n-1}Q_n - (a_nQ_n - P_n)^2$
$\therefore Q_{n+1} = Q_{n-1} + 2a_nP_n - a_n^2Q_n = Q_{n-1} + a_n(P_n - P_{n+1})$ [B]
[A],[B] let us calculate the series where we have
$P_2^2 + Q_2Q_1 = R$
$Q_2 = (R - (a_1Q_1 - P_1)^2)/Q_1 = (R - P_1^2)/Q_1 + 2a_1P_1 - a_1^2Q_1$

And if you think about it, this shows that since by hypothesis $(R - P^2)/Q \in \mathbf{Z}$ then $\forall i$, P_i, $Q_i \in \mathbf{Z}$.

If we begin with

$$\frac{P_1 + \sqrt{R}}{Q_1} = \frac{p_{n-1}x_n + p_{n-2}}{q_{n-1}x_n + q_{n-2}} = \frac{p_{n-1}P_n + p_{n-2}Q_n + p_{n-1}\sqrt{R}}{q_{n-1}P_n + q_{n-2}Q_n + q_{n-1}\sqrt{R}} \qquad [1]$$

we can derive

$$q_{n-1}P_n - q_{n-2}Q_n = Q_1p_{n-1} - P_1q_{n-1}$$

and using P_1 from LHT and MT above

$$q_{n-1}P_n + q_{n-2}Q_n = ((-1)^{n-1}Q_1)/(q_{n-1}x_n + q_{n-2}) + q_{n-1}\sqrt{R}$$

and continuing back from [1], we get to

$$(-1)^{n-1}P_n = P_1(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + (R - P_1^2)/Q_1 \cdot q_{n-1}q_{n-2} - Q_1p_{n-1}q_{n-2}$$
$$(-1)^{n-1}Q_n = -2p_{n-1}q_{n-1}P_1 + (R - P_1^2)/Q_1 \cdot q_{n-1}^2 + Q_1p_{n-1}^2$$

all of which gives us another way to calculate our series. Now, long story short, if you drive all of this to the explicit formulae for P_n, Q_n, $\sqrt{R} - P_n$ and $2\sqrt{R} - Q_n$ we can know the following, all of which is proved by the derivation:

Thm. 5.5. For some value of $n = \nu$ and all $n > \nu$, P_n, Q_n, $\sqrt{R} - P_n$ and $2\sqrt{R} - Q_n$ are all positive and $P_n < \sqrt{R}$ and $Q_n < 2\sqrt{R}$.

Cor. 1. $P_n, Q_n \in \mathbf{N}$ ∴ after $n = \nu$, P_n cannot have more than $\sqrt{R}$ different values and Q_n more than $2\sqrt{R}$ ∴ $x_n = (P_n + \sqrt{R})/Q_n$ cannot have more than $\sqrt{R} \cdot 2\sqrt{R} = 2R$ values ∴ the cycle of recurrence has at most 2R steps and in this cycle P_n, Q_n are always positive.

Cor. 2. It follows that $n>\nu \Rightarrow A_n < 2\sqrt{R}$ ∴ none of the partial quotients in the cycle can exceed the max integer in $2\sqrt{R}$.

Cor. 3. Ultimately, we have $P_n + Q_n > \sqrt{R}$

Cor. 4. And ultimately $P_n + Q_{n+1} > \sqrt{R}$

Cor. 5. $\sqrt{R} > P_m \Rightarrow$ by Cor. 3, 4: $P_m - P_n < Q_n < Q_{n-1}$

Let's go the other way. Every RCF is a simple quadratic surd.
Consider a RCF with period $a_1 - a_r$: $x_1 = a_1: a_2, a_3, \cdots, a_r, \ldots_R$
Let p/q. p'/q' be the last two convergents before the value of $a_1: a_2, a_3, \cdots, a_r$

$$\therefore \quad x_1 = a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +} \cdots \frac{1}{a_r +}\frac{1}{x_1} = \frac{px_1 + p'}{qx_1 + q'}$$

$$\therefore \quad qx_1^2 + (q' - p)x_1 - p' = 0 \text{ where } x_1 \text{ is the positive root}$$

$$\therefore \quad x_1 = (p - q' + \sqrt{((p - q')^2 + 4pq)})/2q = (L + \sqrt{N})/M$$

Here, $a_1 \neq 0$, $p/q > 1$ ∴ $p > q > q'$ and this surd cannot take the form $\sqrt{N}/M$.

Example

Evaluate 1:2, 1, 1, ...
Last two convergents to 1 + 1/2 + 1/1 are 3/2 and 4/3
∴ $x_1 = (4x_1 + 3)/(3x_1 + 2)$
∴ $3x_1^2 - 2x_1 - 3 = 0$ ∴ $x_1 = (1 + \sqrt{10})/3$

Thm. 5.6. The Form of an RCF = √N/M

The square root of $q \in \mathbf{Q}$ or $\sqrt{(C/D)}$ can be put in the form $\sqrt{N}/M$ where $N = CD$, $M = D$. And its RCF with period α_i [1-s] must take form:

$$x_1 = a_1: a_2, \ldots, a_r, \alpha_1, \alpha_2, \ldots, \alpha_s, \ldots_R \qquad [1]$$

We can use our earlier RCF formula by letting $P_1 = 0$, $r = N$, $Q_1 = M$. Let P'/Q', P/Q be last two convergents before the quotient leading up to a_r and p'/q', p/q be the last two before α_s. If we let $y_1 = \alpha_1: \alpha_2, \ldots, \alpha_s$ then

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +}\frac{1}{y_1} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +}\frac{1}{\alpha_1 +} \cdots \frac{1}{\alpha_s +}\frac{1}{y_1}$$

$$\therefore \quad x_1 = \frac{Py_1 + P'}{Qy_1 + Q'} = \frac{py_1 + p'}{qy_1 + q'} \qquad [2]$$

[cont'd]

Eliminate y and

$$(Qq' - Q'q)x_1^2 - (Qp' - Q'p + Pq' - P'q)x_1 + (Pp' - P'p) = 0 \qquad [3]$$

If $x_1 = \sqrt{N}/M$ we need

$$M^2 x_1^2 - N = 0 \qquad [4]$$

But for [3] and [4] to be true, we need

$$Qp' - Q'p + Pq' - P'q = 0 \qquad [5]$$

and

$$(Pp' - P'p)/(Qq' - Q'q) = -N/M^2 \qquad [6]$$

But LHS [6] =

$$P'p'/Q'q' \cdot (P/P' - p/p')/(Q/Q' - q/q') \qquad [7]$$

where, if you work them out, the num and denom of the RHT = $a_r - \alpha_s$ + some proper fraction. So [6] cannot be satisfied. Therefore there can be **only one** partial quotient in the acyclic part of [1].

$$\therefore\ x_1 = a: \alpha_1, ..., \alpha_s, ...$$
$$= a + \frac{1}{\alpha_1 +} \cdots \frac{1}{+ \alpha_s +} \frac{1}{1/(1/(x_1 - a))}. \qquad [8]$$

$$\therefore x_1 = (p/(x_1 - a) + p')/(q(x_1 - a) + q')$$

$$\therefore q'x_1^2 - (p' + q'a - q)x_1 - (p - ap') = 0 \qquad [9]$$

$$\therefore x_1 = (p' + q'a - q)/2q' + (\sqrt{(p' + q'a - q)^2 + 4(p - ap')q'})/2q' \qquad [10]$$

So for $x_1 = \sqrt{N}/M$ RHT num = 0 [11]
and $q'^2 \cdot N/M^2 = (p - ap')q$ ■ [12]

Cor. 1.
By [11] $p'/q' + a = q/q'$
$\therefore 2a: \alpha_1, ..., \alpha_{s-1} = \alpha_s: \alpha_{s-1}, ..., \alpha_1$
$\therefore \alpha_s = 2a \quad \alpha_{s-1} = \alpha_1 \quad \alpha_{s-2} = \alpha_2 \quad ... \quad \alpha_1 = \alpha_{s-1}$
$\therefore \sqrt{N}/M = a: \alpha_1, \alpha_2, ... \alpha_s, \alpha_1, 2a, \alpha_1, ...$
Cor. 2. Also from [11] $q'^2 N/M^2 = pq' - p'(q - p')$
$\therefore q'^2 N/M^2 = p'^2 = pq' - p'q = \pm 1$
with upper/lower sign for even/odd convergent

So our results for $(P_1 - \sqrt{R})/Q_1$ apply here to $\sqrt{N}/M$:

$a_1 = a$ $\quad x_1 = (P_1 + \sqrt{R})/Q_1 = (0 + \sqrt{N})/M$
$a_2 = \alpha_1$ $\quad x_2 = (P_2 + \sqrt{R})/Q_2 = (L_1 + \sqrt{N})/M_1$
$a_3 = \alpha_2$ $\quad x_3 = (P_3 + \sqrt{R})/Q_3 = (L_2 + \sqrt{N})/M_2$
… $\quad$ …
$a_s = \alpha_{s-1}$ $\quad x_s = (P_s + \sqrt{R})/Q_s = (L_{s-1} + \sqrt{N})/M_{s-1}$
$a_{s+1} = 2a$
$a_{s+2} = \alpha_1$

$\therefore L_n = a_{n-1}M_{n-1} - L_{n-1}$ and when $n = 1$ $L_1= aM$ [1]
$Ln^2 + M_{n-1}M_n = N$ [2]
$L_1 + MM_1 = N$
$(-1)^n L_n = (N/M)q_nq_{n-1} - Mp_np_{n-1}$ [3]
$(-1)^n M_n = Mp_n^2 - (N/M)qn^2$ [4]

It follows that no $L_i > \sqrt{N}$ and no partial quotient or $M_i > 2\sqrt{N}$. We know $\forall L_i,M_i > 0$ and the L_i,M_i form cycles collateral with the cycle of partial and total quotients. The M_i are reciprocal and the L_i are too after the first term (i.e. elements eqD from ends are equal.)

So if $L_m = L_{n+1}$ $\quad M_m = M_n$ $\quad \alpha_m = \alpha_n$
then $L_{m-1} = L_{n+2}$ $\quad M_{m-1} = M_{n+1}$ $\quad \alpha_{m-1} = \alpha_{n+1}$
And if m = n this becomes $L_n = L_{n+1} \therefore L_{n-1} = L_{n+2}$ $\quad M_{n-1} = M_{n+1}$ $\quad \alpha_{n-1} = \alpha_{n+1}$
So if two consecutive L_i are equal they are the middle terms in an even cycle and the partial quotient and L_i corresponding to M_1,M_2 will be the middle terms of their respective odd cycles. And if $M_n = M_{n+1}$ and $\alpha_n = \alpha_{n+1}$ they too are the middle terms of even cycles and the L_{n+1} is then the middle term of its odd cycle. All of which amounts to the reduction of any calculation of these elements by half, right?

Example

$$\frac{\sqrt{8463}}{39} = 2 + \frac{-78 + \sqrt{8463}}{39} = 2 + \frac{1}{(78 + \sqrt{8463})/61}$$
$$RHT = 2 + \frac{-44 + \sqrt{8463}}{61} = 2 + \frac{1}{(44 + \sqrt{8463})/107}$$
$$RHT = 1 + \frac{-63 + \sqrt{8463}}{107} = 1 + \frac{1}{(63 + \sqrt{8463})/42}$$
$$RHT = 2 + \frac{-63 + \sqrt{8463}}{42} = 3 + \frac{1}{(63 + \sqrt{8463})/42}$$
RHT = 1 + …

Partial Quotients:	2	1	3	1	2	4
L_i (rational dividends):	78	44	63	63	44	78
M_i (divisors):	39	61	107	42	107	61

$\therefore \sqrt{8463}/39 = 2: 2, 1, 3, 1, 2, 4, \ldots_R$

Diophantine Problems

We consider eqns or systems of eqns which are indeterminate and we seek only solns in **N**. For eqns 1° of x,y we need consider only

$$ax \pm by = c$$

where p(a,b) because if a,b not prime to each other, there are no solns in **N**. So let's find all the integral solns in ax - by = c and then sort out those which are in **N**. We can find a particular soln ∈ **Z** of

$$ax - by = c \qquad [1]$$

Because p(a,b), CF a/b has itself as last convergent. Let penultimate convergent be p/q then

$$aq - pb = \pm 1 \qquad [2]$$

∴ $a(\pm cq) - b(\pm cp) = c$ [3]

∴ $x = \pm cq \quad y = \pm cp$ is a soln [4]

Let x,y be any such soln ∈ **Z** then from [1]-[3]

$$a(x - (\pm cq)) - b(t - (\pm cp)) = 0$$

∴ $(x - (\pm cq))/(y - (\pm cp)) = b/a$ [5]

∴ $\exists t \in \mathbf{Z}: (x - (\pm cq) = bt) \wedge (y - (\pm cp) = at)$

∴ $x = \pm cq + bt \quad y = \pm cp + at$ is general soln [6]

where sign is taken as in [2]. If a/b > p/q upper sign in [2] and

$$x = cq + bt \quad y = cp + at \qquad [6']$$

So there are infinite solns ∈ **Z**. To get solns ∈ **N**,

$$-cp/a \le t \le +\infty$$

So there are infinite solns in **N**. If a/b < p/q, Sym. x = -cq + bt and y = -cp + at and again there are infinite solns in **N**. For our next entertainment we have

$$ax + by = c \qquad [7]$$

and there's always a soln in **Z,** for with the same p/q we have

$$(\pm cq)a + (\mp cp)b = c \qquad [8]$$

and $x = \pm cq$, $y = \mp cp$ is a soln ∈ **Z** and Sym. to the above all solns in **Z** are

$$x = \pm cq - bt \quad y = \mp cp + at$$

To get solns in **N** let a/b > p/q then cp/a < cq/b and then with cp/a ≤ t ≤ cq/b we have

$$x = cq - bt \quad y = -cp + at \qquad [9]$$

And Sym. if $a/b < p/q$ then $x = -cq - bt \quad y = cp + at$. Here the number of solns cannot exceed 1 + mod(cq/b - cp/a) or as (aq - pb) = 1 the solns in **N** cannot exceed 1 + c/ab.

Example

Req. solns in **N** for $8x + 13y = 159$
$8/13 = 0$: 1, 1, 1, 1, 2 with penultimate convergent 3/5
$\therefore 8\cdot5 - 13\cdot3 = 1$
$8(795) + 13(-477) = 159$ or $(+159\cdot5)8 + (-159\cdot3)13 = 159$
$\therefore$ general soln: $x = 795 - 13t \quad y = -477 + 8t$
where $795/13 \geq t \geq 477/8$ or t can only be 60 or 61
$\therefore$ all solns in **N** are (15,3) and (2,11)

And for an interesting bit of thinking, you can reconcile those solns to the idea of geometric points. If you are curious, you will find that this method above can be used for a single equation of more than two variables like $3x + 2y + 3z = 8$. Now let's consider systems like

$$ax + by + cz = d \qquad [1]$$
$$a'x + b'y + c'z = d' \qquad [2]$$

where all coeffs are in **Z**. This system is equivalent to

$$-(ca')x + (bc')y = (dc') \qquad [3]$$
$$ax + by + cz = d \qquad [4]$$

where (ca') is our old friend ca' - c'a. Let gcm((ac'),(bc')) = δ. Then if (dc') !divby δ there are no solns in **Z**. Else soln takes form

$$x = x'' + (bc')t/\delta \quad y = y'' + (ca')t/\delta \qquad [5]$$

Where (x",y") is any soln of [3] in **Z** and $t = \forall t \in$ **Z**. Sub [5] → [4]

$$cz - c(ab')t/\delta = d - ax'' - by'' \qquad [6]$$

where the coeff of t is in **Z**. If there is a soln of [6] in **Z** ⇒ z = z', t = t'

$$\therefore \qquad (z - z')/(t - t') = (ab')/\delta$$

If ε = gcm(δ,(ab')) then

$$z = z' + (ab')u/\varepsilon \quad t = t' + \delta u/\varepsilon \qquad [7]$$

where $u = \forall u \in$ **Z** and now we have

$$x = x' + (bc')u/\varepsilon \quad y = y' + (ca')u/\varepsilon \quad z = z' + (ab')u/\varepsilon \quad [8]$$

where $x' = x'' + (bc')t'/\delta$ $y' = y'' + (ca')t'/\delta$. In [8] let u = 0 then x=x', y=y', z=z' and (x',y',z') is soln in **Z** of system [1],[2]. So [8] gives all the solns in **Z** of [1],[2] where (x',y',z') is any particular soln: $\varepsilon = gcm((bc'),(ca'),(ab'))$ and $u = \forall u \in \mathbf{Z}$. To get solns in **N**, you simply limit u.

Example

$3x + 4y + 27z = 34$
$3x + 5y + 21z = 29$
$(bc') = -51$ $(ca') = 18$ $(ab') = 3$ $\varepsilon = 3$
One soln is (1,1,1) ∴ general soln: $x = 1 - 17u$ $y = 1 + 6u$ $z = 1 + u$
Only soln in **N** is (1,1,1)

Now we can consider second degree Diophantine equations. If p_n/q_n is nth convergent and M_n = (n+1)th rational divisor in $\sqrt{(C/D)}$ then

$$Dp_n^2 - Cq_n^2 = (-1)M_n \qquad [1]$$

∴ in $Dx^2 - Cy^2 = +H$ where $C,D,H \in \mathbf{N}$ and if C/D not a perfect square there are infinite solns of RHS and the same holds for $Dx^2 - Cy^2 = -H$. A useful case has D = 1

$$x^2 - Cy^2 = \pm H$$

where $C,H \in \mathbf{N}$ and C not a perfect square. Then we need ±H as a $(-1)^n M_n$ in the development of $\sqrt{C}$ as a SCF for infinite solns. It follows that $x^2 - Cy^2 = 1$ where C is positive and not a perfect square has infinite solns. If quotients in the period of $\sqrt{C}$ are even = 2s then $(-1)^{2s}M_{2s} = 1$ and with $t = \forall t \in \mathbf{N}$

$$p_{2ts}^2 - Cq_{2ts} = +1$$

and all solns are

$$x = p_{2ts} \quad y = q_{2ts} \qquad [A]$$

If quotients are odd in number = 2s-1 then $(-1)^{4s-2}M_{4s-2}$ and $(-1)^{8s-4}M_{8s-4}$ both equal unity and solns are

$$x = p_{4ts-2t} \quad y = q_{4ts-2t} \qquad [B]$$

Sym. with an odd number of quotients in the development of $\sqrt{C}$ then $x^2 - Cy^2 = -1$ has infinite solns. In

$$x^2 - Cy^2 = \pm H \qquad [1]$$

we can limit ourselves to p(x,y) ∴ p(x,y,H) and if $H < \sqrt{C}$ then all solns of [1] are provided by the convergents of $\sqrt{C}$ as above. This means that if (p,q) is a soln then p/q is a convergent of $\sqrt{C}$. You should be able to show

$$p/q - \sqrt{C} < 1/(q^2(p/q\sqrt{C} + 1)) \quad [2]$$
$$p/q - \sqrt{C} < 1/2q^2 \quad [3]$$

So all solns of

$$x^2 - Cy^2 = 1 \quad [4]$$

come from penultimate convergents in successive or alternate periods of √C and if the number of quotients in the period of √C is even then

$$x^2 - Cy^2 = -1 \quad [5]$$

has no integral soln. If odd, then solns are from penultimate convergents of the alternate periods of √C. If (p,q) first soln of [4], then the general soln is

$$x + y\sqrt{C} = (p + q\sqrt{C})^n \quad [6]$$
$$x - y\sqrt{C} = (p - q\sqrt{C})^n$$

and for [5]

$$x + y\sqrt{C} = (p + q\sqrt{C})^{2n-1} \quad [7]$$
$$x - y\sqrt{C} = (p - q\sqrt{C})^{2n-1}$$

Example

Solns ∈ **Z** for $x^2 - 13y^2 = 1$
$\sqrt{13} = 3: 1, 1, 1, 1, 6, \ldots_R$
Taking the 10th convergent where p'q - pq' = +1: x = 649 y = 180
$649^2 - 13\cdot180^2 = 1$
∴ general soln: $x = ½((649 + 180\sqrt{13})^n + (649 - 180\sqrt{13})^n)$
$y = ½((649 + 180\sqrt{13})^n - (649 - 180\sqrt{13})^n)/\sqrt{13}$

And now ask yourself where you have seen that form before.

Consider $x^2 - Cy^2 = \pm H$ where C not a perfect square, H pos. and >√C. I will give an example which you may be able to completely follow. If not, let your curiosity lead you on. It is at this point the the theory of these solns begins to get hairy.

Example

Req. solns of $x^2 - 15y^2 = 61$
Let $(K_1^2 - 15)/61 = H_1$ where K_1 is just an arbitrary integer thing ≤ ½H ∴ ≤ 30
∴ $K_1^2 = 15 + 61H_1$
So we need a perfect square in 15 + 61(n): n ∈ (0, 1, 2, ...) : $K_1^2 < 30^2 = 900$
And the only one is 625 ∴ $K_1 = 25$ $H_1 = 10$
Because $H_1 > \sqrt{15}$ we must repeat the process
$(K_2^2 - 15)/10 = H_2$ where $K_2 \leq 5$ ∴ $K_2^2 = 5$ ∴ $K_2 = 5$ $H_2 = 1$

This gives us $x_2^2 - 15y_2^2 = 1$ soln (4,1)
$\therefore$ general soln of $x^2 - 15y^2 = 61$ is

$$x_2 = ½((4 + \sqrt{15})^n + (4 - \sqrt{15})^n)$$
$$y_2 = 1/2\sqrt{15} \cdot ((4 + \sqrt{15})^n - (4 - \sqrt{15})^n)$$

$\therefore x_1 = (5x_2 \mp 15y_2)/1 \quad y_1 = (5y_2 \mp x_2)/1$
$\therefore x = (25x_1 \mp 15y_1)/10 \quad y = (25y_1 \mp x_1)/10$
$\therefore x = 14x_2 \mp 45y_2 \qquad y = \mp 3x_2 + 14y_2$

Here's a reason to fill in any blanks you have in understanding that example. We can make the soln of the general second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

depend upon an eqn in the form of $x^2 - Cy^2 = H$ and bring all this to bear on conics and geometry. I leave this to your curiosity.

General Continued Fractions

Our SCF and RCF have only unity in their numerators and positive integers in their denominators. Let's go back to the general form of continued fractions where the nums and denoms of the descending fractions are any elements of **Z** and denote such CF as GCF. For

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}$$

we have as before

$$x_1 = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \qquad [1]$$

which we denote as

$$a_1 \colon b_2/a_2, b_3/a_3, \ldots$$

With nums and denoms in **Z**, we could have

$$1\colon 1/1, 1/\text{-}1, 1/\text{-}1, \ldots_R \qquad [A]$$

where the 3d convergent is 1 + 1/(1-1). So we need a way to deal with such anomalies as arise without throwing out the baby with the bath water. We have two varieties of GCF which we can explore:

$a_1\colon b_2/a_2, b_3/a_3, \ldots$ **first class GCF**
$a_1\colon b_2/\text{-}a_2, b_3/\text{-}a_3, \ldots$ **second class GCF**

and what follow regards the first class but by replacing b_i with $-b_i$ it will all hold for the second class.

Prop. 5.1. If p_1/q_1, p_2/q_2 successive convergents of [1] then

$$p_n = a_np_{n-1} - b_np_{n-2} \qquad [2]$$
$$q_n = anq_{n-1} + b_nq_{n-2} \qquad [3]$$

with initial conditions $p_0 = 1 \quad p_1 = a_1 \quad q_1 = 1 \quad q_2 = a_2$. These definitions of p_n, q_n are used in all cases, especially those like [A] above which present difficulties. They define the sense of a GCF in all cases where sense exists.
Cor. 1. In a 1st class GCF where $p_n,q_n > 0$ and $a_n \geq 1$ then p_n,q_n monotonically increase with n. This holds for 2d class GCF where $a_n \geq 1 + b_n$. From all this, it does not follow that $Lp_n = \infty$ or that $Lq_n = \infty$ which you can prove using

$$p_n - p_{n-1} = (a_n -1)p_{n-1} + b_np_{n-2} \qquad [4]$$

Cor. 2. $$\frac{p_n}{p_{n-1}} = a_n + \frac{b_n}{a_{n-1}} + \frac{b_{n-1}}{a_{n-2}} + \cdots + \frac{b_2}{a_1} \qquad [5]$$
$$\frac{q_n}{q_{n-1}} = a_n + \frac{b_n}{a_{n-1}} + \frac{b_{n-1}}{a_{n-2}} + \cdots + \frac{b_3}{a_2} \qquad [6]$$

Prop. 5.2. From [2],[3] $p_nq_{n-1} - p_{n-1}q_n = (-1)^n \prod b_i\ [2\text{-}n]$ [1]
Cor. 1. Convergents by [2],[3] are not necessarily in lowest terms.
Cor. 2. $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n \prod b_i\ [2\text{-}n]}{q_nq_{n-1}}$ [2]
Cor. 3. $\frac{p_n}{q_n} = a_1 + \frac{b_2}{q_1q_2} + \frac{b_2b_3}{q_2q_3} + \cdots + \frac{(-1)^n \prod b_i\ [2\text{-}n]}{q_{n-1}q_n}$ [3]
Cor. 4. $p_nq_{n-2} - p_{n-2}q_n = (-1)^{n-1}\, a_n\prod b_i\ [2\text{-}(n\text{-}1)]$ [4]
$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1} \prod b_i\ [2\text{-}(n\text{-}1)]}{q_nq_{n-2}}$ [5]
Cor. 5. $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \div \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} = \frac{-b_nq_{n-2}}{q_n} = \frac{-b_nq_{n-2}}{a_nq_{n-1} + b_nq_{n-2}}$. [6]
Cor. 6. If 1st class GCF convergents are odd/even in number they are monotonically increasing/decreasing. Every odd/even convergent is less/greater than all successive convergents. A 2d class GCF where $\forall n$, $a_n \geq 1 + b_n$ the convergents are greater than zero and create an increasing series.

Our p_n, q_n belong to a class of rational ifns in a_i,b_i. Our p_n is determined by

$$p_2 = a_2p_1 + b_2p_0$$
$$p_3 = a_3p_2 + b_3p_1$$
$$\ldots$$
$$p_n = a_np_{n-1} + b_np_{n-2}$$

where initial state is $p_0 = 1 \quad p_1 = a_1$ and q_n are similarly determined by

$q_3 = a_3q_2 + b_3q_1$
$q_4 = a_4q_3 + b_4q_2$
…
$q_n = a_nq_{n-1} + b_nq_{n-2}$

and initial state is $q_1 = 1 \quad q_2 = a_2$. Now, keep in mind that we have just described $p_i(a_j,b_k)$ and $q_i(a_j,b_k)$ as functions. Our q_n is the same fn of a_i [2-n] and b_i [3-n] as p_n is of a_i [1-n], b_i [2-n]. Euler created a notation for this which, so far as Chrystal takes us, is not a method of calculation but a convenience of stating relations. Euler's notation:

$$p_n = K \left| \begin{matrix} & b_2, \ldots, b_n \\ a_1, & a_2, \ldots, a_n \end{matrix} \right| \; [3] \qquad q_n = K \left| \begin{matrix} & b_3, \ldots, b_n \\ a_2, & a_3, \ldots, a_n \end{matrix} \right| \; [4]$$

where the vertical bars "|" should be a single large parentheses. The p_n in this notation are called **continuants** of order n. When all numerators are unity we have a **simple continuant**

$$p_n = K(a_1, \ldots, a_n)$$

where in all this the K(), with its big parens, is a functional notation like f() or φ().
If r < s,

$$K(r,s) = K \left| \begin{matrix} & b_{r+1}, \ldots, b_s \\ a_r, & a_{r+1}, \ldots, a_s \end{matrix} \right| \qquad [5]$$
$$K(s,r) = K \left| \begin{matrix} & b_s, \ldots, b_{r+1} \\ a_s, & a_{s-1}, \ldots, a_r \end{matrix} \right| \qquad [5]$$
$$K(r,r) = a_r \therefore K(1,1) = p_1 = a_1 \quad p_0 = q_1 = 1 = K()$$

Prop. 5.3. A continuant of order n is an ifn of n° of its constituents:

$$K(l,n) = a_nK(l,n-1) + b_nK(lm\ n-2)$$
$$K(l,n-1) = a_{n-1}K(l,n-2) + b_{n-1}K(l,n-3)$$
$$\ldots \qquad [7]$$
$$K(l, l+1) = a_{l+1}K(l,l) + b_{l+1}K()$$
$$K(l,l) = a_l$$
$$K() = 1$$

The law here is

$$K(1,n) = a_nK(1,n-1) + b_nK(1,n-2) \qquad [8]$$

If you care to pursue this, Hindenburg created an algorithm to produce continuants. But for simplicity's sake, we will only look at Euler's rule:

Write down the first term which is $\prod a_i$ [1-n]. To get the rest, omit from this product in every possible way (combinatorics!) *one or more pair of successive a_i always replacing the second of the pair by a b_i of the same order.*

Oh, yeah. Simple. Perhaps an example is in order:

Example

Terms of K(1,4)   1st term = $a_1a_2a_3a_4$
Omit a_1a_2, a_2a_3, a_3a_4 replacing each with b_2, b_3, b_4 and we get $b_2a_3a_4$, $a_1b_3a_4$, $a_1a_2b_4$. Omit two pair for b_2b_4. And that's all, folks.

Back to our Prop. 5.3.
Cor. 1. Values of a continuant are not altered by reversing their order:

$$K\begin{vmatrix} b_2, \ldots, b_n \\ a_1, a_2, \ldots, a_n \end{vmatrix} = K\begin{vmatrix} b_n, \ldots, b_2 \\ a_n, a_{n-1}, \ldots, a_1 \end{vmatrix}$$

or K(l,m) = K(m,l)

In this notation, [1],[2] of Prop. 2 become:

$$K(1,n)K(2,n-1) = K(1,n-1)K(2,n) = (-1)^n \prod_{1-n} b_i K()K()$$
$$K(1,n)k(2,n-2) = K(1,n-2)K(2,n) = (-1)^n \prod_{2-(n-1)} K()K(n,n)$$

And these fall under

Thm. Euler's Continuant Theorem
$K(1,n)K(l,m) - K(1,m)K(l,n) = (-1)^{m-l+1} \prod_{l-(m+1)} K(1,l-2)K(m+2,n)$

Round about here, without a bunch of exercises, we're getting in over our heads. Let's look at a couple of uses for the above and then move on to shallower waters. From Euler's Theorem, these follow

$$K(a_1,\ldots a_i,a_i \ldots a_1)^2 = K(a_1,\ldots, a_i)^2 + K(a_1,\ldots, a_{i-1})^2$$
$$K(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i-1}, \ldots, a_1) = K(a_1, \ldots, a_{i-1})(K(a_1, \ldots, a_i) + K(a_1, \ldots, a_{i-2})$$

which allow him to show that every prime p of form $4\lambda + 1$ is the sum of two integer squares.

Example

$13 = 3 \cdot 4 + 1$
13/1 = 13; 13/2 = 6+1/2; 13/3 = 4+1/3; 13/4 = 3+1/4; 13/5 = 2:1,1,2; 13/6 = 2+1/6
∴ 13 = K(13) = K(6,2) = K(4,3) = K(3,4) = K(2,1,1,2) = K(2,6)
∴ $13 = K(2,1,1,2) = K(2,1)^2 = K(2)^2 = 3^2 + 2^2$

When $K(1,n) = a_nK(1,n-1) + b_nK(1,n-2) \equiv p_n = a_np_{n-1} + b_np_{n-2}$ can be solved as a finite difference equation, we can derive an expression for our GCF where we are actually finding the general term of a series.

Example

Required: the nth convergent to F = 1: 1,1,1,1,...
We must solve $p_n = p_{n-1} + p_{n-2}$ where $p_0 = 1$ and $p_1 = 1$
$\therefore K(1,n) = p_n = ((1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1})/2^{n+1}\sqrt{5}$ [1]

$$\therefore \frac{p_n}{q_n} = \frac{K(1,n)}{K(2,n)} = \frac{\text{RHS [1]}}{((1+\sqrt{5})^n - (1-\sqrt{5})^n)/)2^n\sqrt{5})} = \tfrac{1}{2}\frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n)}$$

Here we also see that the number of different terms in a continuant order n is also:

$$\text{RHS [1]} = 1/2^n \cdot (C_{n+1|1} + C_{n+1|3} + C_{n+1|5} + \cdots)$$

(Combinatorics!)

Convergence of Infinite Continued Fractions

Here we are talking about $L_\infty p_n/q_n$ which can converge or oscillate or diverge to $\pm\infty$. Our 1: 1/1, 1/-1, 1/-1, 1/-1, ... oscillates between 1, 0, $-\infty$ and our 1: 1/1, 1/1, 1/1, ... converges to $-\frac{1}{2} + \frac{1}{2}\sqrt{5}$ $\therefore$ 1: 1/-($-\frac{1}{2} + \frac{1}{2}\sqrt{5}$), 1/-1, 1/-1, 1/-1, ... diverges to $-\infty$. Divergence of a GCF is different from divergence of a series. Here, if we dropped the second term, this last GCF would converge. In this analysis, we retain our restriction of examining the same two classes of GCF.

From what we know of the two sequences forms , a 1st class GCF cannot ber divergent. $Lp_m/q_m = A$ $Lp_{2n-1}/q_{2n-1} = B$, A,B finite with $A \geq B$. If A = B the CGF converges, else oscillates. Further, a 1st class GCF is convergent if

1) $\sum a_{n-1} \cdot a_n/b_n$ converges; or
2) $L\ a_{n-1}\ a_n/b_n > 0$; or
3) $L\ a_n/b_n > 0 \wedge \sum a_n$ diverges; or
4) $L\ a_{n+1}b_n\ /\ a_{n-1}b_{n+1} > 1$.

Thm. 5.7. If a first class GCF is put in the form $d_1: d_2, d_3, \ldots$:
$d_1 = a_1$, $d_2 = a_2/b_2$, $d_3 = a_3b_2/b_3$, $d_4 = a_4b_3/b_4b_2$, ..., $d_n = a_nb_{n-1}b_{n-3}\ldots/b_nb_{n-2}\ldots \Rightarrow$
it converges if one of the two following series converges and oscillates if they both do.
1) $d_3 + d_5 + d_7 + \cdots$ 2) $d_2 + d_4 + d_6 + \cdots$

Examples
1) 1: $1^2/2$, $2^2/2$, $3^2/2$,...

$$\therefore d_{2n+1} = \frac{2(2n-1)^2(2n-3)^2\cdots 3^2 1^2}{(2n)^2(2n-2)^2\cdots 4^2 2^2}$$

By Stirling's Thm, as $n\to\infty$ $d_{2n+1} \cong 2(2n!)/2^{4n}(n!)^4$
$= 2((\sqrt{(2\pi 2n)}(2n/\varepsilon)^{2n})/(2^{2n}(2\pi n)(n/\varepsilon)^{2n}))^2 = 2/\pi n$
$\therefore \sum d_{2n+1}$ comparable to $\sum 1/n$ $\therefore$ GCF converges

2) 0: 1/2, 2/3, 3/4, ...
$L\ a_{n-1}a_n/b_n = L(n-1)n/(n+1) = \infty$ $\therefore$ GCF converges

There is no comprehensive law for convergence of a 2d class GCF. But we do have:

Thm. 5.8. If an infinite GCF has form

$$F = 0{:}b_2/\text{-}a_2,\ b_3/\text{-}a_3,\ \ldots,\ b_n/\text{-}a_n,\ \ldots :$$

$\forall n,\ a_n > b_n + 1 \Rightarrow$ GCF converges to the finite limit $F \le 1$. If a_n ever greater than $b_n + 1$, $F<1$. If $\forall n,\ a_n = b_n + 1$ then

$$F = 1/(1 + b_2 + b_2b_3 + \cdots + \prod_{2\text{-}n} b_i + \cdots$$

so that F equals/is less than unity as that denom is div/cnv. Note by this form, $\forall b_i > 0$.
Cor. 1. If the b_i are any quantity whatever, then the denom =

$$0{:}\ 1/1,\ b_2/\text{-}b_2{+}1,\ b_3/\text{-}b_3{+}1,\ \ldots,\ b_n/\text{-}b_n{+}1$$

And we get a theorem of Euler's, who was a very busy man, by letting u_i = ith term of the denom. $\therefore$ $1 + u_1 + u_2 + \cdots + u_n$ =

$$0{:}\ 1/1,\ u_1/\text{-}1{+}u_1,\ u_2/\text{-}u_1{+}u_2,\ u_1u_3/\text{-}u_2{+}u_3,\ \ldots,\ u_{n-3}u_{n-1}/\text{-}u_{n-2}{+}u_{n-1},\ u_{n-2}u_n/\text{-}u_{n-1}u_n,\ \ldots$$

Let's look at a few more useful ideas concerning GCF before we dive into number theory. Here are two theorems by Lagrange:

Thm. 5.9. $\forall i,\ a_i,\ b_i\ [1\text{-}n] \in \mathbf{N} \Rightarrow 0{:}b_2/a_2,\ b_3/a_3,\ \ldots,\ b_n/a_n$ converges to $r \in$ **R-Q** if after some $n > \nu$, $(a_n > b_n{+}1) \wedge (a_n = b_n{+}1$ only a finite number of times).

Thm. 5.10. With same a_i,b_i then $0{:}\ b_2/\text{-}a_2,\ b_3/\text{-}a_3,\ \ldots,\ b_n/\text{-}a_n$ converges to some $r \in$ **R-Q** if after some $n > \nu$, $(a_n > b_n{+}1) \wedge (a_n = b_n{+}1$ only a finite number of times).

Finally, let's convert the series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ into its equivalent GCF of the second class:

$$0{:}\ b_1/a_1,\ b_2/\text{-}a_2,\ \ldots,\ b_n/\text{-}a_n \qquad [1]$$

Def. A GCF is **equivalent to a series** when its nth convergent equals the sum of the first n terms of the series.

Since the convergents are given, we can leave the denoms q_i arbitrary and let $q_0 = 1$. For [1] we have

$$p_n/q_n - p_{n-1}/q_{n-1} = \prod_{1\text{-}n} b_i/q_{n-1}q_n \qquad [2]$$
$$q_1 = a_1 \quad q_2 = a_2q_2 - b_2 \quad \cdots \quad q_n = a_nq_{n-1} - b_nq_{n-2} \qquad [3]$$
$$p_1/q_1 = b_1/q_1 \qquad [4]$$
$$p_n/q_n = u_1 + u_2 + \cdots \qquad [5]$$

From [2],[5]:

[6]

$u_n = \prod_{1\text{-}n} b_i / q_{n-1} q_n$
$u_{n-1} = \prod_{1\text{-}(n-1)} b_i / q_{n-2} q_{n-1}$
…
$u_2 = b_1 b_2 / q_1 q_2$
$u_1 = b_1 / q_1$

From [6] using successive pairs

$$b_1 = q_1 u_1,\ b_2 = q_2 u_2 / u_1,\ \ldots,\ b_n = q_n u_n / q_{n-2} u_{n-1} \quad [7]$$

From [3],[7]

$$a_1 = q_1,\ a_2 = q_2(u_1 + u_2)/q_1 u_1,\ \ldots,\ a_n = q_n(u_{n-1} + u_n)/q_{n-1} u_{n-1} \quad [8]$$

$$\therefore\ S_n = \sum_{1\text{-}n} u_i = \frac{q_1 u_1}{q_1 -} \frac{q_2 u_2 / u_1}{q_2(u_1+u_2)/q_1 u_1 -} \cdots - \frac{q_n u_n / q_{n-2} u_{n-1}}{q_n(u_{n-1}+u_n)/q_{n-1}u_{n-1}}. \quad [9]$$

If you are clever, you can see that we can factor our all the q_i here and we are left with

$$S_n = 0: u_1/1, (u_2/u_1)/\text{-}((u_1+u_2)/u_1), (u_3/u_2)/\text{-}((u_2+u_3)/u_2), \ldots \quad [10]$$

$$= \frac{u_1}{1 -} \frac{u_2}{u_1+u_2 -} \frac{u_1 u_3}{u_2+u_3 -} \cdots - \frac{u_{n-2} u_n}{u_{n-1}+u_n} \quad [11]$$

And again that was brought to you by Euler. By giving u_i different forms we can do this:

$$v_1 x + v_2 x^2 + \cdots + v_n x^n = \frac{v_1 x}{1 -} \frac{v_2^2 x}{v_1+v_2 x -} \cdots - \frac{v_{n-1}^2 x}{v_{n-1}+v_n x}$$

$$x/v_1 + x^2/v_2 + \cdots + x^n/v_n = \frac{x}{v_1 -} \frac{v_1^2 x}{v_1 x+v_2 -} \frac{v_2^2 x}{v_2 x+v_3 -} \cdots - \frac{v_{n-1}^2 x}{v_{n-1}x+v_n}$$

And with all this mightiness, let's square the circle like Brouncker did:

If $x \in (-\pi/4, \pi/4) \Rightarrow \tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$

$$= \frac{x}{1 +} \frac{1^2 x^2}{3\text{-}x^2 +} \frac{3^2 x^2}{5\text{-}3x^2 +} \frac{5^2 x^2}{7\text{-}5x^2 +} \cdots$$

$\therefore\ \pi/4 = 1: 1^2/2, 3^2/2, 5^2/2, \ldots$

which will square your circle if you use four of them.

6. Number Theory

We had a decent introduction to number theory from De Morgan in DME. But we've seen that Chrystal's viewpoint is often very different from De Morgan's. So let's take it from the top.

Prop. 6.1. M,N,p,q,r ∈ **N** If M = pm+r N = qm+r then M,N are **congruent modulus m**, denoted M≡N(mod m) or M≡N if m understood.
Cor. 1. If M≡N(mod m) then M,N differ by a multiple of m, i.e. M = N + pm, p∈**Z**
Cor. 2. If M≡N and one has a factor of some n∈**N** then so does the other
OR from the opposite viewpoint, p(M,n) ⇔ p(N,n)

Prop. 6.2. For ∀m∈**N** all integers can be arranged in m groups mod m, i.e. mod 0,1,2,…,m-1 OR consecutive integers divided by m produce remainders which are a cyclic perm of n integers 0 - (m-1)

Example

A perfect cube has the form of either 7p or 7p±1 (p prime)
Proof
∀n∈**N** has form 7m, 7m±1, 7m±2, or 7m±3 mod 7
Also $(7m \pm r)^3 = (7m)^3 \pm 3(7m)^2r + 3(7m)r^2 \pm r^3$
$= (7^2m^3 \pm 21m^2r + 3mr^2)7 \pm r^3$
$= M7 \pm r^3$
∴ 4 cases:
$N^3 = (7m)^3 = (7^2m^3)7$
$N^3 = (7m \pm 1)^3 = M7 \pm 1$
$N^3 = (7m \pm 2)^3 = M7 \pm 8 = (M \pm 1)7 \pm 1$
$N^3 = (7m \pm 3)^3 = (M \pm 4)7 \mp 1$ ■

If that seemed fun, I really encourage you to get a text like Chrystal's in PDF and check out the exercises in number theory. If you will get the basic ideas here into your head, the exercises are very satisfying and actually fun. And they reveal a great deal about the form of number.

Thm. 6.1. $f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$, $x \equiv r \pmod m \Rightarrow f(x) \equiv f(r) \pmod m$
Proof
$x = qm + r$
$(qm + r)^n = (qm)^n + C_{n|1}(qm)^{n-1}r + \cdots + C_{n|n-1}(qm)r^{n-1} + r^n$
$= (q^nm^{n-1} + C_{n|1}q^{n-1}m^{n-2}r + \cdots + C_{n|n-1}qr^{n-1})m + r^n$
$= M_nm + r^n$
Sym. $(qm + r)^{n-1} = M_{n-1}m + r^{n-1}$
…
∴ $x = qm + r \Rightarrow f(x) = p_0 + p_1r + p_2r^2 + \cdots + p_nr^n + (p_1M_1 + \cdots + p_nM_n)m$
$= f(r) + Mm$ ■

Cor. 1. To test divby of f(x) by m for ∀x∈**Z** we need only test f(x) for x∈{0,1,...,m-1}

Example

When is f(x) = x(x+1)(2x+1) divby 6?
f(0) = 0; f(1) = 6; f(2) = 30; f(3) = 84; f(4) = 180; f(5) = 330
All divby 6 ∴ f(x) always divby 6

Cor. 2. f(qf(r) + r) is always divby f(r) as f(qf(r) +r) = Mf(r) + f(r) = (M+1)f(r)
Note: it follows from this that no ifn can produce only prime numbers.

Example

Show x^4-1 divby 5 ⇔ p(x,5)
z∈**Z** = 0,±1,±2 (mod 5)
f(0) = -1; f(1) = 0; f(2) = 15
0,15 divby 5 -1 !divby 5 ∴ thm follows

As a sidebar, note that we can also use a previous idea to test f(x) for divby.
Let $f_n(x)$ be an n° ifn of x. Then,

$$\begin{aligned} f_n(x+1) - f_n(x) = p_0 &+ p_1(x+1) + \cdots + p_{n-1}(x+1)^{n-1} + p_n(x+1)^n \\ - p_0 &- p_1x - \cdots \quad - p_{n-1}x^{n-1} \quad - p_nx^n \end{aligned} \qquad [1]$$

So x^n disappears and [1] is (n-1)°. If we divide by m, we have

$$(f_n(x+1) - f_n(x))/m = f_{n-1}(x)/m$$

Then, by inspection, if RHS divby m, LHS divby m. And if $f_n(0)$ divby m then $f_n(1)$ is too. And if nothing is obvious at this point, we can do with f_{n-1} what we did with f_n and wash, rinse, repeat until divby is laid bare to inspection.

Example

$f_5(x) = x^5 - x$ is always divby 5
Proof
$f_5(x+1) - f(x) = (x+1)^5 - (x+1) - x^5 + x = 5x^4 + 10x^3 + 10x^2 + 5x = M5$
$f_5(1) = 0$ ∴ $f_5(2) - f_5(1) = M_05$ ∴ $f_5(2) = M_05$
Sym. $f_5(3) - f_5(2) = M_15$ ∴ $f_5(3) = (M_0 + M_1)5$ and so on ■

We know from DME that any composite (non-prime) number has a factor not greater than its square root. It follows that:

Prop. 6.2. The divisors of $N = a^\alpha b^\beta c^\gamma \cdots$ must have form $a^{\alpha'} b^{\beta'} c^{\gamma'} \cdots$ where the power of any factor, say α' is an integer in [0,α]. So the divisors of N are the terms of

$$\begin{aligned}&(1 + a + a^2 + \cdots + a^\alpha)\\ &\times (1 + b + b^2 + \cdots + b^\beta)\\ &\times (1 + c + c^2 + \cdots + c^\gamma)\\ &\times \cdots \qquad [1]\end{aligned}$$

Cor. 1. Each of these terms has the form $(a^{\alpha+1} - 1)/(a-1)$. So the sum of the divisors is

$$\frac{(a^{\alpha+1}-1)(b^{\beta+1}-1)(c^{\gamma+1}-1)\cdots}{(a-1)(b-1)(c-1)\cdots}$$

Cor. 2. And if in [1], we let a,b,c,... = 1 each term in the product is unity and the sum becomes the number of divisors. So the number of divisors is $(\alpha+1)(\beta+1)(\gamma+1)\cdots$

Cor. 3. And the number of ways the N can be expressed as two factors is

$$\tfrac{1}{2}(1 + (\alpha+1)(\beta+1)\cdots) \text{ or } \tfrac{1}{2}(\alpha+1)(\beta+1)\cdots$$

as N is or is not a perfect square.

Cor. 4. The number of ways N can have two factors prime to each other is 2^{n-1} where n is the number of prime factors of a^α, b^β, c^γ, ...

Def. We use **φ(N)** to denote the number of integers, including 1, which are less than, and prime to $N \in \mathbf{N}$ (or, I suppose, **Z**).

Thm. 6.2. Euler's φ(N) Theorem

If $N = a_1^{\alpha 1} a_2^{\alpha 2} a_3^{\alpha 3} \cdots a_n^{\alpha n}$ then $\varphi(N) = N(1 - 1/a_1)(1 - 1/a_2)\cdots(1 - a_n)$

Proof

Let's find all the numbers that do share factors with N.

The multiples of $a_i < N$ are $1a_i, 2a_i, \cdots (N/a_i)a_i$ ∴ these are N/a_i in number

Sym. those of $a_ia_j < N$ are $1a_ia_j, 2a_ia_j, \cdots, (N/a_ia_j)a_ia_j$ ∴ N/a_ia_j in number and so on.

So consider

$$\begin{aligned}&N/a_1 + N/a_2 + \cdots\\ &- N/a_1a_2 - N/a_1a_3 - N/a_1a_4 - \cdots\\ &+ N/a_1a_2a_3 + N/a_1a_2a_4 + \cdots\\ &- N/a_1a_2a_3a_4 - \cdots\\ &+ \cdots \qquad [2]\end{aligned}$$

There are $C_{n|1}$ terms in the first line, $C_{n|2}$ (combinatorics!) in the second line, and so on. Now consider every multiple of r factors $a_1a_2a_3\cdots a_r < N$. This will appear in the first line $C_{r|1}$ times, in the second line $C_{r|2}$ times and (again) so on. So it appears

$$C_{r|1} - C_{r|2} + C_{r|3} - \cdots \pm C_{r|r-1} \mp C_{r|r} = 1 - (1 - 1)^r = 1 \text{ time in } [2]$$

Then every number without repetition or omission which has a factor in common with

N appears one time in [2]. So [2] can be written

$$N - N(1 - 1/a_1)(1 - 1/a_2)(1 - 1/a_3)\cdots(1 - 1/a_n) \qquad [3]$$

Therefore numbers which don't share factors with N, and we call these "prime to N" are N-[3]:

$$\varphi(N) = N(1 - 1/a_1)(1 - 1/a_2)(1 - 1/a_3)\cdots(1 - 1/a_n) \; \blacksquare$$

Example

$\varphi(100)$: $100 = 2^2 5^2 \;\therefore \varphi(100) = 2^2 5^2(1 - 1/2)(1 - 1/5) = 40$

Thm. 6.3. $M = PQ \wedge p(P,Q) \Rightarrow \varphi(M) = \varphi(P)\varphi(Q)$
Proof
Stand on your head and prove it yourself.
Cor. 1. P,Q,R,S,... prime to each other $\Rightarrow \varphi(PQRS\ldots) = \varphi(P)\varphi(Q)\varphi(R)\varphi(S)\ldots$

Thm. 6.4. All divisors of $N \in \{d_i\}$ [1-?] $\Rightarrow \sum\varphi d_i = N$ [1]
Proof
Each d_i is a term in the product of [1] in Prop. 6.2.
$\therefore d_i = a_1^{\alpha 1'} a_2^{\alpha 2'}\cdots \;\therefore\; \varphi(d_i) = \varphi(a_1^{\alpha 1'} a_2^{\alpha 2'}\cdots) = \varphi(a_1^{\alpha 1'})\varphi(a_2^{\alpha 2'})\cdots$ as $\forall a_i$ are primes
$\therefore \sum d_i = (1 + \varphi(a_1) + \varphi(a_1^2) + \cdots + \varphi(a_1)^{\alpha 1})$
$\quad \times (1 + \varphi(a_2) + \varphi(a_2^2) + \cdots + \varphi(a_2)^{\alpha 2}) \times \cdots$ [2]
$\varphi(a_1^r) = a_1^r(1 - 1 - 1/a_1) = a_1^r - a_1^{r-1}$
$\therefore$ each term in [2] has form $= 1 + (a_1 - 1) + (a_1^2 - a_1) + \cdots + (a_1^{\alpha 1} - a_1^{\alpha 1-1}) = a_1^{\alpha 1}$
$\therefore [2] = a_1^{\alpha 1} a_2^{\alpha 2}\cdots = N \;\blacksquare$ (And all that without any combinatorics!)

Def. Given any fraction a/b, **I(a/b)** is the integral part of a/b or 0 if a/b is a proper fraction.

Thm. 6.5. The highest power of prime p which divides m! exactly is $I(m/p) + I(m/p^2) + I(m/p^3) + \cdots$ until p^n is max n: $p^n \leq m$.
Proof
The numbers in 1,2,3,...,m divby p are 1p, 2p, 3p, ... kp where k max int: $kp \leq M$.
So k = I(m/p) Sym. for p^2, p^3, ... $\therefore$ As p is prime, the thm holds. $\blacksquare$

Examples

1) Highest power of 7 dividing 1000! exactly.
Repeatedly dividing 1000 and its integral remainders by 7 we get 142, 20, 2.
$140 + 20 + 2 = 164 \;\therefore 7^{164}$ required.

2) Required prime factors of 25!

Primes < 25 are

2	3	5	7	11	13	17	19	23	and successive quotients are
12	8	5	3	2	1	1	1	1	
6	2	1							
3									
1	and these sum to								
22	10	6	3	...					

∴ $25! = 2^{22}3^{10}5^{6}7^{3}\cdots$

3) Required: highest power of 5 dividing $27\cdot 28\cdot 29\cdots 100$

Ans. 5^{18} (Finally, an exercise! Bon chance!)

Thm. 6.6. If $f + g + h + \cdots \leq m \Rightarrow m!/f!g!h!\cdots \in \mathbf{N}$

Proof

This is nothing but perms in combinatorics(!). Or you can chase down a proof that uses the last theorem. Or make your own using the last theorem. You're a free agent here. Act like one.

Cor. 1. If $f + g + h + \cdots \leq m \wedge$ none of f, g, h, ... equal m (like it would?) then $m!/f!g!h!\cdots$ is divby m if m is prime.

Cor. 2. The product of r successive integers is divby r!

Here is a bit of a teaser. Fermat challenged Wallis and the English mathematicians with this problem which in Chrystal's graduated problem set is number 9 out of 21. So you definitely have the mojo for it. *Find a cube the sum of whose divisors is a square.* This shows Fermat's dominion over the form of number with respect to the ideas we have so far seen in this chapter.

Thm. 6.7. Given the AP: k, k + a, k + 2a, ..., k + (m-1)a and dividing each by m: p(a,m), the least remainders are a perm of 0,1,2,3,...,(m-1)

Proof

All remainders are different. Else

$$k + ra = \mu m + \rho \qquad k + sa = \mu' m + \rho$$

∴ $(r - s)a = (\mu - \mu')m$

∴ $(r - s)a/m = \mu - \mu'$ ↴ because p(a,m) r,s < m ∴ only m possible remainders ■

Cor. 1. If remainders of k,a wrt m = k',a' remainders occur as k', k'+a', k'+2a', ..., k'+ra' which last equals or exceeds m and must be reduced by m and a' added again for the remainders.

Example

k = 11 a = 25 m = 7 ∴ S_n = 11, 36, 61, 86, 111, 136, 161

∴ k' = 4 a' = 4 ∴ remainders 4, 4+4-7 = 1, 5, 5+4-7 = 2 ... or 4, 1, 5, 2, 6, 3, 0

Cor. 2. Beyond m terms, remainders repeat in the same order.

Cor. 3. The number of terms in the series of remainders prime to m is $\varphi(m)$

Cor. 4. In the series of remainders 0,1,2,...,(m-1), let those prime to m be $r_1, r_2, \ldots, r_n$ where $n = \varphi(m) \Rightarrow$ the numbers $k + r_1a, k + r_2a, \ldots$ are all prime to m, where k = 0 or is a multiple of m and p(m,a). And their remainders divided by m are a perm of r_i

Thm. 6.8. As with last theorem but !p(m,a), gcm(a,m) = g: a = ga' m = gm' ⇒ remainders of k, k + a, …, k + (m-1)a wrt m recur in a shorter cycle of m'

Example

k = 11 a = 25 m = 15
S_n = 11, 36, 61, … g = 5 a' = 5 m' = 3
a" = 2 k" = 11 ga" = 10 ∴ remainders are 11, 6, 1, 11, 6, 1, …

Cor. 1. If gcm(a,m') = g divides k exactly or k = 0 then the remainders of k, k + a, … are 0g, 1g, 2g, …, (m' - 1)g in a constant perm.

And from all this we derive

Thm. 6.9. Fermat's Theorem
If m is prime, p(a,n) ⇒ (a^{m-1}- 1) divby m
Proof
The proof in the text is about 15 short lines long and uses the necessary form of a, 2a, ..., (m-1)a. Give it a try if you are enjoying number theory. Or look it up. Or skip it. Do what you like -- I'm not your mom.

Fermat's theorem can be restated as **m prime, p(a,m) ⇒ (a^m - a) divby m**. Then Euler generalizes it as **a,m∈N, p(a,m) ⇒ $a^{\varphi(m)}$- 1 divby m**. Euler also defines **allied numbers** as: a,b,m ∈ **N**, ab ≡ 1 (mod m) and he develops this idea by showing that from 1,2,3,…,(m-1) we can exhaust 2,3,...,(m-2) by combining them into allied pairs. And this leads to

Thm. Wilson's Theorem
m prime ⇒ ((m - 1)! + 1) divby m
Cor. 1. m !prime ⇒ ((m - 1)! + 1) !divby m

Then Lagrange makes Fermat and Wilson special cases of:

Thm. Lagrange's Theorem
p prime ∧ (x + 1)(x + 2)(x + 3)⋯(x + p - 1) = x^{p-1}+ A_1x^{p-2}+ ⋯ + A_{p-2}x + A_{p-1} ⇒ ∀A_i divby p.
Proof (combinatorics!)
(x + p)(RHS) = (x + 1)($(x+1)^{p-1}$+ $A_1(x+1)^{p-2}$+ ⋯ + A_{p-2}(x+1) + A_{p-1})
∴ px^{p-1}+pA_1x^{p-2}+ ⋯ + pA_{p-2}x + pA_{p-1}) = ($(x+1)^p$ - x^p) + A_1($(x+1)^{p-1}$ - x^{p-1}) + ⋯
∴ pA_1 = $C_{p|2}$ + $c_{p-1|1}A_1$
pA_2 = $C_{p|3}$ + $C_{p-1|1}A_1$ + $C_{p-2|1}A_2$
⋯
p prime ∴ $C_{p-1|1}$, $C_{p-2|1}$, $C_{p-3|1}$, … ! divby p
∴ ∀A_i divby p ■
Cor. 1. x = 1 ⇒ 2·3⋯p = 1 + (A_1 + ⋯ + A_{p-2}) + A_{p-1}
∴ A_{p-1} + 1 ≡ ((p-1)! + 1) divby p (Wilson)

Cor. 2. $x^p - x = x(x+1)\cdots(x+p-1) - (1 + A_{p-1})x - (A_1x^{p-1} + \cdots + A_{p-2}x^2)$
LHS is p consecutive integers and divby p
p prime ∴ $1 + A_{p-1}$ divby p
∴ p prime ⇒ $x^p - x$ divby p
(If p(x,p) then Fermat)

What follows will complete our look into number theory. The following is, to my mind, a pure abstract algebra, as continuants were another such algebra. These are algebras, not of computations, but of relations which extend our understanding of their subject. If you will play with either a bit using concrete examples, you will see that this is so. You will recall De Morgan's short demonstration of the partition of an integer from DME. We partition them in two classes.

Class	**Notation**	**Part's Relation**
1st	P(n|p|q)	parts equal or unequal
2d	P_u(n|p|q)	parts unequal

Examples

P(n|p|q)	number n, p parts, max part = q
P(n|p|≤q)	number n, p parts, all parts ≤ q
P_u(n|≤p|*)	number n, p or fewer unequal parts, parts of any size
P_u(n|p|odd)	number n, p parts, each an odd integer
P(n|*|1,2,2^2,2^3,...)	number n, any number of parts, each part a power of 2

If you pursue this, you will find that partition of integers relate to series as developed by the ubiquitous Euler and to invariant theory as developed by Sylvester (not the cat). To give you an idea of their usage consider these, which will be Greek to most of those who do not take a LHS they have dominion over and figure out how it relates to the RHS below:

$$(1+zx)(1+zx^2)\cdots(1+zx^q) = 1 + \sum P_u(n|p|\leq q)z^px^n$$
$$(1+zx)(1+zx^2)\cdots(1+zx^q)/(1-z) = 1 + \sum P_u(n|\leq p|\leq q)z^px^n$$
$$(1+x)(1+x^2)\cdots(1+x^q) = 1 + \sum P_u(n|*|\leq q)x^n$$
$$(1+zx)(1+zx^2)\cdots = 1 + \sum P_u(n|p*)z^px^n$$
$$(1+x)(1+x^2)\cdots = 1 + \sum P_u(n|*|*)x^n$$

Chrystal's chapters on generalized trigonometric series (which I am holding back for Second Circle of Trigonometry) contain many theorems which can be directly deduced from the above and similar identities. We can develop a formula for enumerating partitions of n into any number of parts less than q and create a table of P(n|*|≤q):

$1/(1-x)(1-x^2)\cdots(1-x^q) = 1 + \sum P(n|*|\leq q)x^n$
Multiply both sides by $(1-x^q)$ ∴ (take a deep breath ...)
$1 + \sum P(n|*|\leq(q-1))x^n = 1 + \sum(P(n|*|\leq q) - P(n-q|*|\leq q))x^n$ where $P(0|*|\leq q) = 1$
∴ for n≥q, $P(n|*|\leq q) = P(n|*|\leq(q-1)) + P(n-q|*|\leq q)$
and if n<q, $P(n|*|\leq q) = P(n|*|\leq(q-1))$

And we get this table:

A \ *n*	1	2	3	4	5	6 *b*	7 *c*	8	9	10	11	12	13	14	15	16	17	18	19	20	C
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	D
2	.	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	
3	.	.	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	
4	.	.	.	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84	94	108	
5	.	.	.	.	7	10	13	18	23	30	37	47	57	70	84	101	119	141	164	192	
6	.	.	.	.	.	11	14	20	26	35	44	58	71	90	110	136	163	199	235	282	
q 7	.	.	.	.	.	.	15	21	28	38	49	65	82	105	131	164	201	248	300	364	
8	.	.	.	.	.	.	.	22	29	40	52	70	89	116	146	186	230	288	352	434	
9	.	.	.	.	.	.	.	.	30	41	54	73	94	123	157	201	252	318	393	488	
10	.	.	.	.	.	.	.	.	.	42	55	75	97	128	164	212	267	340	423	530	
11	.	.	.	.	.	.	.	.	.	.	56	76	99	131	169	219	278	355	445	560	

B E *a* F *d*

Let me just hint at this table's construction, leaving its full understanding to your curiosity. The table is symmetrical. So row 4 col 5 = 6 = row 5 col 4. The first row is P(1|*|≤1), P(2|*|≤1), ... which are all ones. Then, unless you are going to code this and generate it digitally, you use a piece of paper in the form of abcd as shown. So to get the 23 in row 4 col 10, we add the 14 on ab at row 6 col 7 to the 9 on row 4 col 6. So in row 9 the 156 is 146+11. To find, using this table, a value of P(n|p|*) we have:

$$1 + \sum P(n|p|*)x^n z^p = 1/(1-zx)(1-zx^2)\cdots = 1 + \sum x^p z^p/(1-x)(1-x^2)\cdots(1-x^p)$$
$$\therefore \sum P(n|p|*)x^n = \sum x^p/(1-x)(1-x^2)\cdots(1-x^p) = \sum P(n|*|\le p)x^{n+p}$$
$$\therefore P(n|p|*) = P(n-p|*|\le p)$$
$$\text{Sym. } P_u(n|p|*) = P(n-\tfrac{1}{2}p(p+1)|*|\le p)$$

Examples

1) P(20|5|*) = P(15|*|≤5) = 84
2) P_u(20|5|*) = P(5|*|≤5) = 7

It follows that P(n|*|q) = P(n|q|*). And from this we derive

$$P(n|\le p|\le q) = P(n|\le q|\le p)$$
$$P(n-p|q-1|\le p) = P(n-q|p-1|\le q)$$
$$P(n|p|q) = P(n|q|p)$$

And by letting p→∞

$$P(n|\le p|*) = P(n|*|\le p)$$
$$P(n|p|*) = P(n|*|p)$$

Our faces are turning blue. Let's come up for air. Here we come to the end of our look into Chrystal's massive two-volume doorstop of an algebra text. Apart from holding back the three chapters of trigonometric series for a later book, I have entirely skipped two chapters -- Interest (money stuff) and Probability (mortality tables stuff) -- as the metaphysics of death and money are of no interest to me, the latter being combinatorics with a false motive and the former being an unpleasant tedium.

7. Theory of Equations

Equations are expressions of laws and the theory of equations reasons upon these laws. We've already seen quite a bit of equation theory both in DME and in Chrystal's text. With computers sparing us the hell of Horner's method and Galois apparently closing off solns by radicals for eqns of fifth degree and up, the theory of equations has been superceded by numerical analysis and algebraic geometry. I'm not trying to start a fight here. But, after thirty years of programming, computers interest me not at all and algebraic geometry relies upon what Wittgenstein called "amorphous mathematics." I've had my say about amorphism in *Limits of Meaning in Mathematics* which I leave to your curiosity.

But I am a very morphous guy. If Galois (bless his heart and curse those now white-washed sepulchres which caused his untimely despair and suicide) says we can't use soln by radicals, I say, "Fine. What else ya got?" Well, we have the form of number and its basis is law. The law understood will unlock the roots of all equations. That's how mathematics works. Let's see how far the law was brought into our understanding up to the early 20th century when the theory of equations fell from ecclesiastical grace.

The Form of Rational Functions

Recall De Morgan's proof in DME that, given a fn of form [ƒ], x can be so large as to make the first term, x^n, infinitely larger than the remaining terms and so small as to make the term containing x^1 larger than all the preceding terms. Murphy reasons interestingly upon this.

Prop. 7.1. Positive values may be assigned to x such that, for n>m, x^n is greater than any fn φx of form [ƒ] m°.
Proof
$m>n,\ \varphi x = Ax^m + Bx^{m-1} + Cx^{m-2} + \cdots + Px + Q$
$M = \max \in (A,B,C,\ldots,P)$
$\therefore Mx^m \geq Ax^m,\ Mx^{m-1} \geq Bx^{m-1}, \ldots \therefore Mx^m + Mx^{m-1} + \cdots + Mx + Q \geq \varphi x$
(Now recall our algebraic forms)
$\therefore M\cdot(x^{m+1}-1)/(x-1) \geq \varphi x$
$x>1 \therefore M(x^{m+1})/(x-1) \geq \varphi x$
(Here's an interesting part)
$\therefore\ x^n : (Mx^{m+1})/(x-1) :: x^{n-m-1} : M/(x-1)$
As $(x-1)$ or $x \to \infty \Rightarrow M/(x-1) \to 0$
$\therefore x^{n-m-1} = 1 \wedge x^{n-m-1} \to \infty$
$\therefore x^{n-m-1} > M/(x-1)\ \therefore\ x^n > \varphi x$ and infinitely so. ■

Practically speaking (in actual use) if we want $x^{n-m-1} > M/(x-1)$ we can let these be equal and then $x = 1 + M^{1/(n-m)}$ and any x greater than this x makes $x^n > \varphi x$.

Example

$x^3 > 7x^2 + 6x + 5$
$M = 7 \quad n-m = 1 \quad \therefore x \geq 8$

As we work our way through Murphy's *A Treatise on the Theory of Algebraical Equations*, I may indicate that I have skipped some of his ideas which do not fit my idea of the current place of equation theory in mathematics. But I could be wrong about these and you can let your curiosity pursue those as well as any proofs I abbreviate or elide.

Prop. 7.2. Sym. to above, positive values can be assigned to x so small as to make, if n<m, $x^n < \forall\varphi x$ m°.

Murphy's proof is Sym. to Prop 7.1. and you also have De Morgan's from DME.

Prop. 7.3. φ', φ", φ'", …, φ^n 1st, 2d, 3d, …, nth derivatives of φ(x) ⇒
$\varphi(x+h) = \varphi(a) + \varphi'(a)h + \varphi''(a)h^2/2! + \varphi'''(a)h^3/3! + \cdots + \varphi^n(a)h^n/n!$

Murphy doesn't mention Taylor but we already know this as the more common form of Taylor's Series in usage. I'm adding a corollary which you may have noticed already. If not, work out any simple Taylor Series.

Cor.1. ∀φ, n final derivative, $\varphi^n(a)h^n/n! = h^n$

Def. We will denote "**C between A and B**" as **C·|·(A,B)** or did I already mention that?

Prop. 7.4. ∀qfn of x produces a series of values which are nearer each other as the values of x approach each other.

This is the fundamental argument for continuity which we've seen more than once. Murphy proves it by means of Prop. 7.1. used recursively to show that if any term of φ(a+h) vanishes, the remaining terms maintain the proposition because, by the last corollary, they can't all vanish.

Cor. 1. For real values, φ(a)·|·(φ(a+h)+φ(a), φ(a-h)-φ(a))

We have seen in Chrystal how the idea of an ifn's maxima and minima follow from this idea. And this same idea implies the following:

Prop. 7.5. A<C, f(x) ifn of x, f(α) = A, f(γ) = C ⇒ (B·|·(A,C) ⇒ ∃β·|·(α,γ): f(β) = B

Prop. 7.6. ∀qfn f(x) of odd degree n has a real root.

Proof

∃x = α: x^n > f(x) - x^n where x^n is first term of f(x) [Prop. 7.1]

f(α) = A

x = -y ∴ $y^n > ay^{n-1} - by^{n-2} + \cdots > 0$ [Prop. 7.1]

∴ $ay^{n-1} - by^{n-2} + \cdots < 0$

y = β ∴ f(-β) = -B

∴ ∃ γ·|·(α,β): f(γ) = 0 ∧ 0·|·(A,-B) [Prop. 7.5] ■

Cor. 1. If q is the constant term here of f(x) then γ has the opposite sign of q.

Prop. 7.7. ∀qfn(x) of even degree with a negative constant term has two real roots, a,b: a < 0 < b.

Prop. 7.8. Eqns where constant term is very small wrt other coeffs have real roots.

Proof

Given such a φx, divide by the coeff of x ⇒ φx = $-\kappa + x + ax^2 + \cdots = 0$

k < 1/4(M+1) where M again max coeff ⇒ ∃real root < 1/2(M+1), sign opp κ

-κ < 0 ⇒ φ(0) = -κ < 0 ⇒ ∃x < 1: φx = 0 ∴ $-\kappa + \alpha - (M\alpha^2)/(1-\alpha) = 0$

Solving this quadratic: $\alpha = 1/2(M+1) \cdot (1 + \kappa - \sqrt{(1 + 2\kappa + \kappa^2 - 4\kappa(M+1))})$

(4κ(M+1) < 1) ∧ (radical term > $\sqrt{(\kappa^2+2\kappa)}$ > κ)

∴ α < 1/2(M+1) ∧ α∈**R**

∴ f(α) > 0 f(0) < 0 ∴ ∃x∈**R**: x·|·(0,α) ∴ x·|·(0,1/2(M+1))

Sym. κ > 0, ∃root ·|·(0,-1/2(M+1) ■

Prop. 7.9. ∃x: f(x) max or min ⇒ f'(x) = 0

Proof

Assume minimum at f(a) ∴ f(a) < f(a+h) ∧ f(a) < f(a-h). Express the RHTs as Taylor Series and they must have the same sign ∴ f'(a) = 0. Sym. for max. ■

Cor. 1. It follows that f "(a) has the same sign as these same two series.

∴ f"(a) > 0 ⇒ f(a) minimum ∧ f"(a) < 0 ⇒ f(a) maximum

Cor. 2. No qfn of odd degree can have an absolute max or min. They have only **relative** or **local** max or min.

Note that in all these propositions, it is generally true that if the proposition relies on, say, φ'x, but φ'x = 0, then the proposition will still hold using φ"x or whichever further φ^n remains in existence.

Example

φx = x^2 + 5x
φ'x = 2x + 5
φ"x = 2 > 0 ∴ φx has min but no max
Min from φ' = 2x + 5 = 0 ∴ x = -5/2 ∴ min @ (-5/2,-25/4)

Prop. 7.10. ∀qfn even° has absolute min.

Proof

$\varphi x = x^n + ax^{n-1} + bx^{n-2} + \cdots$ M = max value of abs. value of coeffs

Then no x can reduce φx below min of $-Mn$, $-M((n-1)M)^{n-1}$

n even ∴ for ±x, $\varphi x > x^n - M \cdot$ (remaining terms)

1) x < 1 ⇒ $1 + x + x^2 + \cdots + x^{n-1} < n$ ∴ $\varphi x > x_n - Mn$

$x^n > 0$ ∴ $x \in [-1,1] \Rightarrow \varphi x > -Mn$

2) x > 1 ⇒ $nx^{n-1} > x^{n-1} + x^{n-2} + \cdots$ ∴ $\varphi x > x^n - nMx^{n-1}$

Min RHS where RHS' = 0 ∴ min at x = 0 or x = (n-1)M

The 2d gives $x^{n-1}(x - nM) = -Mx^{n-1} = -M((n-1)M)^{n-1}$

∴ $x \in (-\infty,-1) \wedge (1,\infty) \Rightarrow \varphi x > -M((n-1)M)^{n-1}$ ■

Example

$\varphi x = x^2 + ax$ $\varphi' x = 2x + a$ $\varphi'' x = 2$ ∴ abs. (and only) min at $x = -a/2$ ∴ $\varphi x = -a^2/4 = \gamma$

It follows in general that $x^2 + ax = \gamma \equiv x^2 + ax - \gamma = 0$ can have no real roots. The next proposition returns us to the GCM and you can prove it for yourself.

Prop. 7.11. Take φx, $\varphi' x$, $\forall\gamma$. Divide $\varphi x - \gamma$ by $\varphi' x$. Use Euclid's Algorithm to reach a fn ind. of $x \equiv F(\gamma) \Rightarrow$ real roots of $F(\gamma) = 0$ are the max and min values of φx.

Examples

1) $\varphi x = x^2 + ax$ $\varphi' x = 2x + a$

$2(\varphi x - \gamma) = 2x^2 + 2ax - 2\gamma$

$2x + a \,)\, 2x^2 + 2ax - 2\gamma \,(\, x$

$\underline{2x^2 + ax}$

(×a) $ax - 2\gamma \,)\, 2ax + a^2 \,(\, 2$

$\underline{2ax - 4\gamma}$

$a^2 + 4\gamma = 0$ ∴ $\gamma = -a^2/4$ (But you knew that.)

2) $\varphi x = x^3 - 3a^2x + b$

$\varphi' x = 3(x^2 - a^2)$

$\varphi'' x = 6x$

GCM of $x^2 - a^2$ and $x - \gamma$

$x - y \,)\, x^2 - a^2 \,(\, x + \gamma$

$\underline{x^2 - \gamma x}$

$\gamma x - a^2$

$\underline{\gamma x - \gamma^2}$

$\gamma^2 - a^2$ which has two real roots ∴ φx has one min and one max

The next proposition can easily be proved by expanding φ and φ' in Taylor Series and collecting the real and unreal (as "unreal" is shorter than "imaginary" by quite a bit) parts. It is interesting that Murphy brings an $x \in \mathbf{C}$ into play in a real qfn. He does this more than once and I had thought to skip them. But the law of ifns must include complex values and we are unlikely to find this law without venturing onto the complex plane.

It occurs to me that some hopeful, if naive, readers might think I have found this law, the Key to All Polynomial Roots, and am leading up to it. Dream on. I am looking for this law by leaving no tone unsterned. So let me give you this proposition and an example and we'll turn some stones.

Prop. 7.12. f(a) min of qfn f(x) ⇒ for small h,k, f(a + h + ik) < f(a) where sign of h +/- as first derivative $f^n \neq 0$ for x = a has same/contrary sign as $f^{n+1}(x)$. Sym. for f(a) when max of f(x)

What appeals to me about this proposition and example is that, from an age when the phrase "small h,k" usually meant "as small as you please" which aggravated Weierstrass into conniptions, we will see **exactly** how small an h and a k.

Example

Diminish $x^3 - 2x^2 + x - 2$ below its min

$\varphi' = 3x^2 - 4x + 1 \quad \varphi'' = 6x - 4$

$\varphi' = 0$ @ 1, ⅓ $\quad \varphi''(1) = 2 \quad \varphi''(⅓) = -2 \therefore \varphi(1)$ min

$\therefore x = 1 + h + ik \quad \varphi(x) = -2 + (h+ik)^2 + (h+ik)^3$

$= -2 + h^2 + h^3 - k^3 -3hk^2 + i(2h + 3h^2 - k^2)$

unreal part vanishes when $2h + 3h^2 = k^2$

by above, h is positive in $\varphi = -2 + h^2 + h^3 - (2h + 3h^2)(1 + 3h)$

$= -2 - 2h - 8h^2 - 8h^3 < f(1) = -2$

If you play with this next idea, you can prove the proposition for yourself.

Prop. 7.13. ∀qfn f(x), x = a, f and m-1 successive derivatives vanish ⇒ $(x - a)^m$ is a factor of f(x).

Example

$f(x) = x^3 - 5x^2 + 8x - 4 \qquad f(2) = 0$

$f'(x) = 3x^2 - 10x + 8 \qquad f'(2) = 0$

$f''(x) = 6x - 10 \qquad f''(2) = 2$

$\therefore (x - 2)^2$ is a factor of f(x)

We can extend Prop. 7.12:

Prop. 7.14. A monotonically decreasing series of values can be infinitely produced by substituting a series of y + iz for x in a qfn of even°.

Example

$\varphi = x^4 - 4x + 6$

$\varphi' = 4(x^3 - 1) = 0$ @ $x = 1$ ∴ min @ (1,3)

$x = y + iz \Rightarrow \varphi = P + iQ$

$P = y^4 - 6y^2z^3 + z^4 - 4y + 6$ $Q = z(4y^3 - 4yz^2 - 4) = 0$

$Q' = 4z(3y^2 - z^2)$

$z = 1 \Rightarrow$ the two values of y are almost equal. To find exact equality, take gcm($3y^2$-z^2,$3y^2$-$3yz^2$-3) to arrive at an F(z)

$F(z) = 4z^6 - 27 = 0$ for $z = \sqrt{3}/\sqrt[3]{2} \cong 1.3 = a$

Then q is divby $(y-a)^2$ and quotient equated to zero gives a different y by which P is further diminished.

When z = 10 here then $Q = 40(y^3 - 100y - 4)$ or $y \cong 10$

∴ $P = -4(10)^4 - 40 \cdot 6 \cong -4036$ and so on -- to ∞ and beyond.

Prop. 7.15. Any algebraic function, degree n, has exactly n roots.

Murphy in his proof here assumes Gauss without mentioning him. Did you go out and find a nice proof of Gauss's like I told you to? One important thing to keep in mind with these roots is that if a + bi is a root, then so is a - bi.

Roots

Sturm's Theorem was used to find the number and type -- real or unreal -- of an ifn's roots. Now we can do this with a graphing calculator. But that adds nothing to our understanding of the form of number and its laws. Sturm had an insight into the function's derivatives and into Euclid's algorithm. Let's see what we can learn.

Prop. 7.16. Sturm's Theorem

Sturm required the ifn of x to have no multiple roots. But that was shown to be unnecessary. Take any ifn and its first derivative. Then use Euclid's algorithm as if you were finding their GCM but with this difference: change the sign of each remainder before dividing. Let's do an example to refer to throughout:

$f(x) = x^3 - 3x^2 - 4x + 13 = 0$

$f_1 \equiv f'(x) = 3x^2 - 6x - 4$

$f_i(x)$ [2-...] are the successive re-signed remainders denoted the **auxiliary fns**

{f, f_i [2-...]} denoted **Sturm's fns**

$f_2(x) = 2x - 5$

$f_3(x) = 1$

q_i [1-...] ≡ successive quotients. Now here is the heart of what Sturm shows us:

$f(x) = q_1f_1(x) - f_2(x)$

$f_1(x) = q_2f_2(x) - f_3(x)$

$f_2(x) = q_3f_3(x) - f_4(x)$

…

$f_{m-2}(x) = q_{m-1}f_{m-1}(x) - f_m(x)$

Use the example to follow these inferences:

1) The last f_i is either independent of x and cannot vanish if $f(x),f_1(x)$ have a common measure and therefore f(x) has equal roots.

2) Two consecutive auxiliary fns cannot vanish simultaneously unless there are equal roots and then all the following f_i vanish.

3) When any f_i vanishes, its adjacent fns have opposite signs. In the series above you can see that if f_3 vanishes, $f_2 = -f_4$.

4) No alteration of any f_i happens unless x passes through a root of that f_i. When x passes through a root of f(x) one change of sign is lost in the f_i and this cannot happen when x passes through a root of some f_i. You can fiddle with the example to see how this works.

Expand f and f_1 as Taylor Series for f(c+h) and f(c-h) where c is a root of f(x). When x = c-h and h small enough, f has contrary sign to f_1 but with c+h, same sign. So when x goes through c, Sturm's functions lose one change of sign. If you really grasp how this works, just looking at Fourier's Theorem, coming up next, will show you how it fails where Sturm succeeds. By this same method, when some f_i has a root, it shares a sign with one neighbor and is contrary to the other, so passing through some c_i, no change of sign is lost. Back to our example.

Note that if any f_i cannot vanish, it fits Sturm's criterion for being the last f_i and you can stop dividing. Now Sturm was used to find roots. Take our sequence in the example and let x = 0,1,2,3 and record the resulting signs:

	f	f_1	f_2	f_3
0	+	-	-	+
1	+	-	-	+
2	+	-	-	+
3	+	+	+	+

For x = 0,1,2, there are two changes of sign. There are none for x = 3. So there are two positive roots between 2 and 3. You could find them to any degree of approximation by using ever more fine grains of x. Verify that there is one change of sign between -2 and -3 and then plug f into your graphing calculator. We can also use ±∞ to get the number of roots:

	f	f_1	f_2	f_3
-∞	-	+	-	+
+∞	+	+	+	+

But it's not as simple as change of sign, is it? When x = +∞, there are m changes and n-m continuities. Here we have 0 changes and 3 continuities. With -∞ these flip. So when the coeff of the first terms of Sturm's fns are not all positive (which would be zero changes), the excess of changes of -∞ over +∞ is n-2m. So there are n-2m real roots and 2m imaginary, as you can deduce (with a bit of thought) from our last table.

Prop. 7.17. Fourier's Theorem
∀ifn of x, ∀a,b: a < b. Sub a then b into f, f', f", ... ⇒ there cannot be more real roots of f(x) = 0 than the number of changes of signs found in the two substitutions.

A normal text would now go into an analysis of Fourier. But examining Fourier after Sturm is like analysing the earlier square stone wheel that preceded the round one. Let's just look at Fourier's own example and you can see why Sturm is "da man".

$f(x) = x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0$
$f'(x) = 5x^4 - 12x^3 - 72x^2 + 190x - 46$
$f''(x) = 20x^3 - 36x^2 - 144x + 190$
$f^3(x) = 60x^2 - 72x - 144$
$f^4(x) = 120x - 72$
$f^5(x) = 120$

x	**changes of sign**
-10	5
-1	4
0	3
1	3
10	0

Therefore, all real roots between -10 and 10. One between -10 and -1. One between -1 and 0. None between 0,1. And at least one between 1 and 10. But Fourier can't say anything about the last two. Are they between 1 and 10? Are they imaginary? Sturm only knows.

What I learned, working through Murphy, is the importance of φ, φ' and φ". If you will pay close attention to the part these play, from Sturm on, your effort will be surprisingly rewarding.

Prop. 7.18. If the real roots of φx = 0, in descending order of magnitude, are subbed into φ'x, they will alternately produce positive and negative results or, for equal roots, cause φ' to vanish.

You can figure this one out as a little exercise. Let φx have real roots a,b,c in descending order and some unreal ones. Then φx = (x-a)P. Derive φ' where P' comes from P and we have φ'x = (x-a)P' + P. Then sub a,b,c into these factors.. We can note that if all of φ's roots are real then so are all of (φ')'s.

Recall our sigma notation and how it was used with that symmetrical fn stuff.
Let α_i [1-n] be roots of an n° eqn. Let S_i [1-n] be the sums of the powers of these roots. Let a_i [1-n] be the sums of the roots themselves as taken in sigma notation by ones, by twos, ... by n of them. A pattern arises here if we use them to equal nothing.

$S_1 = \sum\alpha_1 \quad a_1 = \sum\alpha_1 \quad \therefore\ S_1 - a_1 = 0$
$S_2 = \sum\alpha_1^2$
But $a_1 \cdot S_1 \neq \sum\alpha_1^2$ as it includes all terms of form $\alpha_1\alpha_2$ appearing twice
$\therefore\ a_1 \cdot S_1 = \sum\alpha_1^2 = 2\sum\alpha_1\alpha_2 \qquad 2a_2 = 2\sum\alpha_1\alpha_2$
$\therefore\ S_2 - a_1S_1 + 2a_2 = 0$
Sym. $S_3 - a_1S_2 + a_2S_1 - 3a_3 = 0$
Then if $m \leq n$, we find
$S_m - a_1S_{m-1} + a_2S_{m-2} - \cdots \pm a_{m-1}S_1 \mp ma_m = 0$
and if $m > n$
$S_m - a_1S_{m-1} + a_2S_{m-2} - \cdots \pm a_nS_{m-n} = 0$

And that was Newton's theorem of the sums of the powers of the roots of an ifn of n°. He could use those to find the coeffs of the ifn if the sums of the powers of the roots were given (which begs the question: Given by whom?) and vice versa. We can get the sums of the powers of the roots in the following way and, amazingly, we might actually want to do so or I wouldn't go on about it.

Prop. 7.19. ∀rifn f(x) of n°. Divide it by its first (highest) term and we have it in the form:

$$f(x) = 1 + P = 0$$

where P has only negative powers of x. Take its natural log

$$\ln(1 + P) = P - P^2/2 + P^3/3 - \cdots \pm P^m/m \cdots$$

Select the coeff of the term with x^{-m} and the coeff of as many of the preceding containing x^{-m}. Their sum is S. Then $-S \cdot m$ is the sum of the mth powers of the roots. Sym. by dividing by the last term, we take the coeff of x^m to get the sum of the inverse powers of x to the mth.

Proof

$f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$
$f(x)/x^n = 1 + P = (1-\alpha_1/x)(1-\alpha_2/x^2)\cdots(1-\alpha_n/x^n)$
$\ln(1 + P) = \ln(1-\alpha_1/x) + \ln(1-\alpha_2/x^2) + \cdots + \ln(1-\alpha_n/x^n)$
$= -\alpha_1/x - 1/2\cdot\alpha_1^2/x^2 - 1/3\cdot\alpha_1^3/x^3 - \cdots - 1/m\cdot\alpha_1^m/x^m$
$- \alpha_2/x - 1/2\cdot\alpha_2^2/x^2 - 1/3\cdot\alpha_2^3/x^3 - \cdots - 1/m\cdot\alpha_2^m/x^m - \cdots$
$= -(S_1/x + 1/2\cdot S_2/x^2 + 1/3\cdot S_3/x^3 + \cdots + 1/m\cdot S_m/x^m \cdots)$
$\therefore -S_m/m$ is the coeff of x^{-m} in $\ln(1+P)$
Sym. divide f(x) by last term which is $-\prod\alpha_i$
$f(x) = 1 + Q = \prod(1 - x/a_i)$ [1-n]
$S_{-1} = \sum 1/a_i$ [1-n]
$S_{-2} = \sum 1/a_i^2$ [1-n]
…
$S_{-m} = \sum 1/a_i^m$ [1-n]
$\therefore\ \ln(1+Q) = -(S_{-1}x + 1/2\cdot S_{-2}x^2 + \cdots + 1/m\cdot S_{-m}x^m + \cdots)$
$\therefore\ S_{-m}/m$ coeff of x^m $\therefore\ S_{-m} = -S_{-m}/m \cdot -m$ ■

If we know the sums of the powers of the roots (or choose such), we can turn this all around the other way:

Prop. 7.20. Expand

$1 - S_1h + S_1^2/2!\cdot h^2 - S_1^3/3!\cdot h^3 + \cdots$ to n+1 terms

$1 - S_2/2\cdot h^2 + S_2^2/1\cdot 2\cdot 2^2 \cdot h^4 - S_2^6/1\cdot 2\cdot 3\cdot 2^3 \cdot h^6 + \cdots$ to terms equal integer above n/2

$1 - S_3^3/3\cdot h^3 + S_3^2/1\cdot 2\cdot 3^3\cdot h^6 - S_3^3/1\cdot 2\cdot 3\cdot 3^3\cdot h^9 \cdots$ to terms equal integer above n/3

$\cdots$

$1 - S_n/n\cdot h^n$

Then in our ifn, the coeff a_m of $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots$ is the sum of the coeffs of h^m above. And because $\ln(1 + a_1/x + a_2/x^2 + \cdots + a_n/x^n) = -S_1/x - ½S_2/x^2 - ⅓S_3/x^3 - \cdots$ we can let $h = 1/x \Rightarrow 1 + a_1h + a_2h^2 + \cdots + a_nh^n = \varepsilon\text{^}S_1h \cdot \varepsilon\text{^}{-½S_2h^2} \cdot \varepsilon\text{^}{-⅓S_3h^3}\cdots$

Examples

1) From all these ideas, find a_n

$a_1 = -S_1 \quad a_2 = S_1^2/1\cdot 2 - S_2/2 \quad a_3 = -S_1^3/3! + S_1S_2/2$ and so on

2) Form an eqn with roots: S_i [1-(n-1)] = 0 $S_n = c$

In this case the first n-1 series above have no powers of h

$\therefore a_1 = 0 \;\; a_2 = 0 \cdots a_{n-1} = 0 \;\; a_n = -S_n/n = -c/n$

$\therefore x^n - c/n = 0$ required.

We can manipulate the heck out of an eqns roots. And these transformations should act as a hint concerning the laws governing roots.

T1 (transformation one) Increasing or decreasing the roots by a constant.
In what follows, if k > 0, we have x - k is, of course, x - k and the roots are decreased by k. If k < 0 we have x - k as x - (-k) or x + k and the roots are increased by k. Given f(x) we want y = x-k and so the required eqn is f(k+y) = 0 which expands as

$f(k) + yf'(k) + y^2/2!\cdot f''(k) + \cdots + y^n/n!\cdot f^n(k) = 0$

$\therefore$ If $f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ [A]

$\Rightarrow f(k+y) = p_0y^n + (p_1 + np_0k)y^{n-1} + (p_2 + (n-1)p_1k + n(n-1)/2!\, p_0k^2)y^{n-2} + \cdots$

$+ (p_r + (n-r+1)p_{r-1}k + \cdots + n(n-1)\cdots(n-r+1)/r!\cdot p_0k^r)y^{n-r} + \cdots + f(k) = 0$

The use of this is the removal of terms in a transformed fn to facilitate the soln by radicals. Above, if $(p_1 + np_0k) = 0$ or $k = -p_1/np_0$ we lose the second term.

Example

Lose the second term of $x^3 - 6x^2 + 4x + 5 = 0$

$p_0 = 1 \quad p_1 = -6 \;\therefore\; k = 2$

$\therefore\; (y+2)^3 - 6(y+2)^2 + 4(y+2) + 5 = 0$

$\therefore y^3 - 8y - 3 = 0$

By observing the signs of successive derivatives in this, we see that when a term is removed from an eqn with all real roots, the adjacent terms have contrary signs and, conversely, if a term is missing in a fn and the "adjacent" terms have same sign there is a pair of imaginary roots.

Another view of this: $f(x) = x^n - ax^{n-1} + bx^{n-2} - \cdots$. Put $x = y + a/n$. Here a is the sum of the roots. The value of x *in the most comprehensive state in which it is capable of algebraic expression, consists of a term free from radical and of other terms affected by them; this simple expression being required to give all the roots, can only do this by varying its radical parts according to the different values of the roots of unity; the part unaffected by any radical is therefore the same in all the roots.* So that part is this a/n, for the radical parts in summation must destroy themselves. If $y = x - a/n$ then values of y have one term fewer than the values of x and which consists essentially of radicals. So this y gives our function a simpler form.

T2 Multiplying the roots by a constant factor

Given any f(x) we can transform it so that its roots are multiplied by a constant factor. Let $y = kx$ then $x = y/k$ and $f(y/k) = 0$ has roots altered by a factor of k. If in [A] above, we put $x = y/k$ and multiply by k^n all that is necessary for the coeffs to be integers is that for each term in the eqn, $p_r k^r y^{n-r}$ every prime factor in the denom of p_r must occur in at least as high a power in k^r. So if f(x) has roots ∈ **Q**, under this condition the transformation has roots ∈ **Z**.

T3 An inversion of powers

Recall reciprocal eqns. If we put $x = 1/y$ and multiply by y^n then if f(x) is missing the mth term from the beginning, f(1/y) is missing the mth term from the other end. So combined with T1, we can remove the reciprocal term from simple, quadratic, cubic, ... eqns.

Now if $y = 1/x$, in a **reciprocal** eqn, if α,β,γ are roots then so are 1/α,1/β,1/γ. If roots are odd in number, one root must be either its own reciprocal or ±1 and can be factored out. Then let $y = x + 1/x$ and reduce the dimensions by half.

Example

$6x^4 + 35x^3 + 62x^2 + 35x + 6 = 0$

$\therefore 6(x^2+1/x^2) + 35(x + 1/x) + 62 = 0$ and now let $x + 1/x = y$

$\therefore 6(y^2-2) + 35y + 62 = 0$ when $y = -5/2, -10/3$

$\therefore$ from first value when $x^2 + 5/2\, x + 1 = 0$ or $x = -2,-1/2$

and then from second, $x^2 + 10/3\, x + 1 = 0$ or $x = -3, -1/3$

Usefulness in mathematics is nice, as we see in these examples. But stopping at usefulness prevents the investigation of underlying law. Whatever we learn of transformation of roots must be applied synergistically to the revelation of law.

T4 Roots as mth powers of roots of given eqn

$\alpha,\beta,\gamma,\ldots$ roots of $f(x) = x^n - ax^{n-1} + bx^{n-2} + \cdots = 0$. $\alpha^m,\beta^m,\gamma^m,\ldots$ transformed roots. $S_1,S_2,\ldots$ sums of powers of roots of f and $\sigma_1,\sigma_2,\ldots$ the same sums for the transformed eqn. Then $\sigma_1 = S_m$, $\sigma_2 = S_{2m}$, $\sigma_3 = S_{3m}$, ... Then if the transformed eqn is

$$f(y) = y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + \cdots$$

the A,B,C,... can be found by Newton's formula of sums of powers of roots above. Since $A = S_m$ and a real root of f(y) is the mth power of a root of f(x) and the second term divided by n is the rational part of the general formula for the roots, then S_m/n is the rational part of the mth power of any root. And this part is available to us in **any** eqn. We can go at all this in reverse.

Examples

1) $\alpha + \beta^{½}$ general root of $x^2 - ax + b = 0$
Rational part of $\alpha + \beta^{½} = \alpha$, of $(\alpha + \beta^{½})^2 = \alpha^2 + \beta$
$S_1 = a$ $S_2 = a^2 + 2b$
$\therefore$ $\alpha = a/2$ $\alpha^2+\beta = (a^2-2b)/2$ $\therefore$ $\beta = a^2/4 - b$

2) $\alpha^{1/3} + \beta^{1/3}$ general root of $x^3 + ax - b$
If $\alpha^{1/3}$ irreducible, so is $\alpha^{2/3}$ $\therefore$ rational part of $\alpha^{1/3} + \beta^{1/3}$ is $2(\alpha\beta)^{1/3}$ $\therefore$ $\alpha\beta$ perfect cube
$S_2 = -2a$ $\therefore$ rational part $= S_2/3$ $\therefore$ $(\alpha\beta)^{1/3} = -a/3$
Rational part of $(\alpha^{1/3} + \beta^{1/3})^2$ is $\alpha+\beta$ since $3\alpha^{2/3}\beta^{1/3} = 3(\alpha\beta)^{1/3}\alpha^{1/3} \in \mathbf{R}-\mathbf{Q}$ as $(\alpha\beta)^{1/3} \in \mathbf{Q}$
$S_3 + aS_1 - 3b = 0$ $\therefore$ $S_3 = 3b$ $\therefore$ $\alpha+\beta = b$
$\therefore$ $\alpha\beta = -a^3/27 - \alpha + \beta = b$
So when $(\alpha\beta)^{1/3} \in \mathbf{Q}$, $\alpha,\beta = b/2 \pm \sqrt{(b^2/4 + a^3/27)}$ where α gets upper sign and β lower

T5 Roots as fns of given fn's roots

Given f(x) with roots $\alpha,\beta,\gamma,\ldots$ and some fn F(x), we want f(y) with roots $F(\alpha),F(\beta),\ldots$
Eliminate x between eqns $f(x) = 0$ and $y - F(x) = 0$ using Euclid's Algorithm and the eqn in y, independent of x, is the required eqn.

OR

Let $F,F',F'',\ldots = 0$ then for $x = 0$, $F(x) = F(0) + F'(0)x + F''(0)x^2/2! + \cdots$
Then, again with the same S_i and σ_i
$\sigma_1 = nF(0) + F'(0)S_1 + F''(0)S_2/2! + \cdots$
$\sigma_2 = n(F(0))^2 + 2F(0)F'(0)S_1 + ((F'(0))^2 + F(0)F'(0))S_2 + \cdots$
and we can use these with Prop. 7.20 to calculate the coeffs of the transformation. Here if the original and the transformed eqns have the same coeff, they have the same roots. So for every real root α there is a root $F(\alpha)$. And if degree is odd, one root is the soln of $F(\alpha) - \alpha = 0$ found by Euclid's Algorithm on f(x), F(x) - x. Factor this out, let $z = F(x)+x$ and the transformed eqn's degree is reduced by half.

T6 Roots as squares of differences of roots

Here's an idea that will come back to haunt us. Let's introduce it using a simple cubic example. This can be done with eqns of any degree, which becomes geometrically hairier as you jack up the degree. Let's transform

$$x^3 + qx + r = 0 \qquad [1]$$

(note the missing 2d term) with roots a,b,c, into another eqn whose roots are the squares of the differences of these. We know:

$$a + b + c = 0 \quad ab + bc + ca = q \quad abc = -r \;\therefore\; a^2 + b^2 + c^2 = -2q$$

The required roots are $(a-b)^2$, $(b-c)^2$, $(c-a)^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$= a^2 + b^2 + c^2 - 2ab - c^2$

$= a^2 + b^2 + c^2 - 2abc/c - c^2$

$= -2q + 2r/c - c^2$ [2]

$\therefore$ when y = 1 and x = c $\Rightarrow y = (a-b)^2$ Sym. for $(b-c)^2$, $(c-a)^2$

So we eliminate x in [1],[2]

$\therefore x^3 + qx + r = 0$

$x^3(2q + y)x - 2r = 0$

$(q + y)x - 3r = 0$

$\therefore x = 3r/(q+y)$ [3]

Sub [3] $\rightarrow$ [1] $\Rightarrow y^3 + 6qy^2 + 9q^2y + 27r^2 + 4q^3 = 0$ [4]

$\therefore 27r^2 + 4q^3 > 0 \Rightarrow$ [4] has a real root > 0 $\therefore$ [1] has two i-roots

$27r^2 + 4q^3 = 0 \Rightarrow$ [4] has one root = 0 $\therefore$ [1] has two equal roots

Now let's do the same thing with

$$x^3 + px^2 + qx + r \qquad [5]$$

$x = x' - p/3 \;\therefore\; (x' - p/3)^3 + p(x' - p/3)^2 + q(x' - p/3) + r = 0$

Or, from T1,

$$x'^3 + q'x' + r' = 0 \qquad [6]$$

$q' = q - p^2/3 \quad r' = 2p^3/27 - pq/3 + r$

Each root of [2] exceeds the corresponding root of [5] by p/3

$\therefore$ the squares of the differences of the roots of [6] are the same as the squares of the differences of the same in [1]. So by the first T6 example the required eqn is

$$y^3 + 6q'y^2 + 9q'^2y + 27r'^2 + 4q'^3 = 0$$

which is

$$y^3 + 2(3q-p^2)y^2 + (3q-p^2)y + [(2p^3-9pq+27r)^2+4(3q-p^2)^3]/27 = 0 \qquad [7]$$

If a,b,c are roots of [1], we have

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = -2(3q - p^2)$$
$$(a-b)^2(b-c)^2 + (b-c)^2(c-a)^2 + (c-a)^2(a-b)^2 = (3q - p^2)^2$$
$$(a-b)^2(b-c)^2(c-a)^2 = -1/27\ [(2p^3 - 9pq + 27r)^2 + 4(3q - p^2)^3]$$

Those last two (and assorted other bits) were from Todhunter's *Theory of Equations* (1880). The Murphy text (1839) was Todhunter's go-to book for some of these ideas. Todhunter also recommended, in a sly way, reading Hargreave, which we will get to after Murphy. Murphy approaches this squares of diffs idea in a more general way. Given [5] for an eqn of any degree, we are to provide the final constant term of its [7].

$$\varphi x = 0,\ n^\circ \text{ with roots } \alpha,\beta,\gamma,\ldots$$
$$\therefore \varphi'x = (x-\beta)(x-\gamma)\cdots + ((x-\alpha)(x-\gamma)\cdots + (x-\alpha)(x-\beta)(x-\delta)\cdots$$
$$\therefore \varphi'(\alpha) = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\cdots$$
$$\varphi(\beta) = (\beta-\alpha)(\beta-\gamma)(\beta-\delta)\cdots$$
$$\text{and so on}$$
$$\therefore \varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma)\cdots = -1^{n(n-1)/2}\ (\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2\cdots$$

and this is the required final term. This product is a sym. fn of the roots. So it can be expressed by means of the coeffs of the given eqn and also by φ'x in the following way:

$$\text{Roots } \varphi'x \text{ are } \alpha_1, \beta_1, \gamma_1, \ldots$$
$$\therefore \varphi'x = n(x-\alpha_1)(x-\beta_1)(x-\gamma_1)\cdots$$
$$\therefore \varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma)\cdots = n^n(\alpha - \alpha_1)(\beta - \alpha_1)(\gamma - \alpha_1)\cdots(\alpha - \beta_1)(\beta - \beta_1)(\beta - \gamma_1)\cdots$$
$$(\alpha - \alpha_1)(\beta - \alpha_1)(\gamma - \alpha_1) = (-1)^n\varphi(a_1)$$
$$(\alpha - \beta_1)(\beta - \beta_1)(\beta - \gamma_1) = -(-1)^n\varphi(\beta_1)$$
$$\cdots \text{ which all leads to}$$
$$\varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma)\cdots = n^n\varphi(\alpha_1)\varphi(\beta_1)\varphi(\gamma_1)\cdots$$

Cubics and Quartics

We go now to cubics and quartics and see even more of how the form of number is revealed by φ' and φ". In order that an eqn have two equal roots, a certain relation of the coeffs is necessary. And this is that the last term in, for example, [1] above, that the last term of [4] above equals 0 as Todhunter just told us.

So for $\varphi x = 0$ to have two equal roots, all roots being $\alpha,\beta,\gamma,\ldots$, we must have $\varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma)\cdots = 0$. And if we call the roots of $\varphi'x = 0$ $\alpha',\beta',\gamma',\ldots$, this is the same as $\varphi(\alpha')\varphi(\beta')\varphi(\gamma')\cdots = 0$, which, for computation, has fewer factors. A quadratic example of this is too easy, even for lazy people like me. Let's do a cubic so I won't be accused of **always** being lazy.

Example

Condition for a cubic to have two equal roots.

$$\varphi x = x^3 + ax^2 + bx + c \qquad \varphi' x = 3x^2 + 2ax + b$$
$$\text{or } (x^2 + 2a/3)(x + b/3) = (x - \alpha')(x - \beta') \text{ as above}$$

We must have

$$(\alpha'^3 + a\alpha'^2 + b\alpha' + c)(\beta'^3 + a\beta'^2 + b\beta' + c) = 0$$
$$= \alpha'^3\beta'^3 + a\alpha'^2\beta'^2(\alpha'+\beta') + b\alpha'\beta'(\alpha'^2+\beta'^2) + c(\alpha'^3+\beta'^3) + a^2\alpha'^2\beta'^2 + ab\alpha'\beta'(\alpha'+\beta') + ac(\alpha'^2+\beta'^2) + b^2\alpha'\beta' + bc(\alpha'+\beta') + c^2 = 0$$

Put b/3 for $\alpha'\beta'$, collect the terms that multiply the same sums of powers of $\alpha'\beta'$ ⇒

$$(10b^3/27 + a^2b^2/9 + c^2) + (4ab^2/9 + bc)(\alpha'+\beta') + (b^2/3 + ac)(\alpha'^2+\beta'^2) + c(\alpha'^3+\beta'^3) = 0 \qquad [5]$$

where $\alpha'+b' = -2a/3 \quad \alpha'^2+\beta'^2 = 4a^2/9 - 2b/3 \quad \alpha'^3+\beta'^3 = -8a^3/27 + 2ab/3$
Sub these → [5] and the required condition is

$$c^2 + 2ac/27\cdot(2a^2 - 9b) + b^2/27\cdot(4b - a^2) = 0 \qquad [6]$$

When c = 0, [4] = $x(x^2 + ax + b) = 0$ and then [6] = $b^2(4b - a^2) = 0$, the RHT being the condition that a quadratic eqn has equal roots. (I slipped that in lazily.) And if a = 0 then [6] = $c^2 + 4b^2/27 = 0$.

This idea we have been playing with provides a general method, using summation and series of natural logs, to eliminate x between $\varphi x + y = 0$ and F(x) when φ,F ifns. I leave this to your curiosity because it is lengthy and if you were not curious, you couldn't stand to read it anyway.

Murphy next treats of solns of cubic eqns. But this is really the study of solns by radicals which only works for eqns of degree less than 5, as we all know by now. This method of soln provide us with the cubic (and quartic) equivalent of

$$x = (-b \pm\sqrt{(b^2 - 4ac)}/2a$$

and if you wish to see these forms derived, knowing the method deadends at degree four, go for it. Here let us see what this dead-ended method says about the form of number. Back in DME, we learned that $x^3 - 1 = 0$ is $x^3 = 1$ and its roots are the cubic roots of unity: 1, ω, ω^2 (these being short-hand for the numerical values). Then ∀k∈**R**, if $x^3 = k$, roots are $1\cdot k^{1/3}$, $\omega k^{1/3}$, $\omega^2 k^{1/3}$ and this works for ∀n∈**N**, $x^n = k$.

The soln by radicals used T1 above to remove the second term of a cubic. So all cubics were reduced to $x^3 + ax + b$. Here x is no longer a simple cube root. it is the sum of two cube roots or

$$x = p^{1/3} + q^{1/3}$$

And by looking at x in this way, we get

$$x^3 = p + 3p^{2/3}q^{1/3} + 3p^{1/3}q^{2/3} + q$$
$$= (p+q) + 3(pq)^{1/3}(p^{1/3} + q^{1/3})$$

This also gives us

$$ax + b = b + a(p^{1/3} + q^{1/3})$$
$$\therefore \quad (p + q + b) + (3(pq)^{1/3} + a)(p^{1/3} + q^{1/3}) = 0$$

Here we have two variables p,q. So we can **force** a zero by requiring, as a second eqn in this system, that

$$(pq)^{1/3} = -a/3$$

You get where that came from, right? So p and q can be determined by

$$p + q = -b$$
$$pq = -a^3/27$$

This is equivalent to

$$p^2 + 2pq + q^2 = b^2$$
$$4pq = -4a^3/27$$

and we end up with

$$p,q = b/2 \pm\sqrt{(b^2/4 + a^3/27)}$$ where p has upper sign and q lower.

We can then use this p and q to say $x = p^{1/3} + q^{1/3}$ and then get an arithmetical value. And we compute on x from that, factor it out of the cubic to get a quadratic, which any high-school student can solve. Due to our forcing fn, our three roots end up being:

$$x_1 = p^{1/3} + q^{1/3} \quad x_2 = \omega p^{1/3} + \omega^2 q^{1/3} \quad x_3 = \omega^2 p^{1/3} + \omega q^{1/3} \qquad [7]$$

Two things strike me as fortuitous in all this. One, that x happens to be the sum of two cube roots, and two, that the forcing of the RHT to zero suffices to put us in a place where we only need to put p+q = -b. "Fortuitous" is perhaps not the right word but, in terms of solutions, it is close. Let's look closer. We determined above that

$$b^2/4 + a^3/27 = 0$$

means that a cubic has two equal roots. So then

$$p = q = b/2 = \sqrt{(-a^3/27)} \;\therefore\; x_1 = 2\sqrt{(-a/3)} \quad x_2 = x_3 = -\sqrt{(a/3)}$$

A few remarks. From [7] $x_1 + x_2 + x_3 = (1 + \omega + \omega^2)(p^{1/3} + q^{1/3})$

But RHS, LHT = 0 ∴ $x_1 + x_2 + x_3 = 0$
By multiplying pairs of roots we derive:

$$x_1x_2 + x_1x_3 + x_2x_3 = 3(\omega + \omega^2)p^{1/3}q^{1/3} = -3(pq)^{1/3}$$

confirming $(pq)^{1/3} = -a/3$. Then

$$x_1x_2x_3 = p + (1+ \omega + \omega^2)(p^{2/3}q^{1/3} + p^{1/3}q^{2/3}) + q = p + q$$

Thus far the form of number. Mathematicians being what the are have pushed all this so far that for any $x^3 + ax^2 + bx + c$ you can get an actual x by plugging things into a formula that fills four lines in a text like Murphy's. They've got one for quartics, too. Very handy if you need an actual x. Not inherently interesting otherwise. Murphy's examination of the quartic shows Simpson's **and** Euler's methods, which are in fact algebraically interesting. But overall his exposition is confined to this dead-ended soln by radicals and extends the above cubic observations of roots and such to the quartic.

But for both cubics and quartics with equal roots, he has an approach to solution that uses φ and φ' which we should look at before we go on. We will look at the cubic version which is more conducive to laziness. Let φx, 3° have two equal roots.

$$\varphi = x^3 + ax^2 + bx + c = 0$$
$$\varphi' = 3x^2 + 2ax + b = 0$$

Eliminate x and arrange in powers of c, and we have, in his notation:

$$[\varphi,\varphi'] = c^2 + 2ac/27\cdot(2a^2 - 9b) + b^2/27(4b - a^2) = 0$$

If we compare [φ,φ'] to the quantity under the √ in the elided long general soln of φ, the latter is ¼[φ,φ'] and you now have a reason to look into that tediously long general soln. We will have quite a bit of investigation into the √ which appears in solutions. So do go look it up. Any algebra text but this one will do.

Now let φ have three equal roots. Then its constant term is a perfect cube and $x = -a/3$. Here, everything under the big $^3\sqrt{}$ in the general soln vanishes and the additional necessary condition for three equal roots, φ" being $6x + 2a$, we have the condition of [φ,φ"] = 0 as:

$$[\varphi,\varphi'] = c - ab/3 - 2a^2/27$$

where the highest power of c is unity. Here, that bit under the √ is ½[φ,φ'] and what we have in this case is

$$x = -a/3 + (½[\varphi,\varphi''] + ½[\varphi,\varphi'])^{1/3} + (½[\varphi,\varphi''] - ½[\varphi,\varphi'])^{1/3}$$

Further, if the roots are α,β,γ, the existence of $\varphi = 0 \wedge \varphi' = 0$ is $\varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma) = 0$ and the coeff of $c^2 = (\alpha\beta\gamma)^2$ here is unity

$\therefore$ $[\varphi,\varphi'] = 1/27\ \varphi'(\alpha)\varphi'(\beta)\varphi'(\gamma)$
$[\varphi,\varphi''] = \varphi''(\alpha)/6 \cdot \varphi''(\beta)/6 \cdot \varphi''(\gamma)/6$

$\varphi'(\alpha) = (\alpha-\beta)(\alpha-\gamma)$ and Sym. for the other two roots.

$\therefore$ $[\varphi,\varphi'] = -1/27\ [(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)]^2$

Further $\varphi''(x)/6 = x + a/3 = x - (\alpha+\beta+\gamma)/3$

$\therefore$ $[\varphi,\varphi''] = 1/27\ (2\alpha - \beta - \gamma)(2\beta - \alpha - \gamma)(2\gamma - \alpha - \beta)$

Sym. for all this with a quadratic

$\varphi = x^2 + ax + b$
$\varphi' = 2x + a$

Eliminate x so that the highest power of b is unity

$\therefore$ $[\varphi,\varphi'] = b - a^2/4$
$x = -a/2 + \sqrt{-[\varphi,\varphi']}$
$[\varphi,\varphi'] = \varphi'(\alpha)/2 \cdot \varphi'(\beta)/2 = -(\beta - \alpha)^2$

If any of this is suggestive to you, Murphy pursues all of these ideas for cubics into quartics. The volume of algebraic computation increases by about a factor of three when you go from cubic to quartic. So I leave these expansions to your curiosity and we can be computationally grateful that soln by radicals is not applicable to quintics and above. I doubt if a supercomputer could, if it were possible, solve much more than a 19° eqn by radicals. Computation is clearly not the point in mathematics.

Roots of Unity

Prop. 7.21. With two binomial eqns $x^a = 1$, $y^b = 1$ and p(a,b) $\Rightarrow$ Unity is the only common root.

Proof

$p(a,b) \Rightarrow \exists A,B \in \mathbf{Z}: aA - bB = \pm 1$
$\therefore a^a = 1 \Rightarrow x^{aA} = 1 \wedge x^b = 1 \Rightarrow x^{bB} = 1$
$\therefore$ if α is common root $\Rightarrow$ by division $\alpha^{aA-bB} = 1$
$\therefore \alpha = 1^{\pm 1} = 1$ is the only possible common root ■

Prop. 7.22. $\alpha \neq 1$ is root of $x^m = 1$, m prime, ∀p·|·(1,m) ⇒ α^p also root fo $x^m = 1$

Proof

$\alpha^m = 1 \therefore \alpha^{p\wedge m} = 1^p = 1 \therefore \alpha^p$ is a root

p,q < m ⇒ α^p, α^q roots

Else $q<p \wedge \alpha^p = \alpha^q \Rightarrow a^{q-p} = 1$

∴ α root of $y^{q-p} = 1$↴ because q<p ∴ p(q,m) ■

Cor. 1. m-1 quantities $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{m-1}$ are roots of $x^m = 1$ and, with unity, these are all the roots.

Cor. 2. m odd, unity the only real root; m even ±1 only real roots

This is a good place to recall, from DME, that the roots of unity $x^m = 1$ are the points on the plane around the origin forming a regular m-gon. This allows you to picture the truth of Cor. 2.

Prop. 7.23. m prime and root of $x^m = 1$ ⇒ ∀qfn of φa is reducible to form

$$A + Ba + Ca^2 + \cdots + Pa^{m-1}$$

Proof

1) If ifn terms of higher power than m-1, it is reducible, i.e. if $a^m = 1 \Rightarrow a^{m+1} = a$ and so on.

2) If φ has form F(a)/φ(a) where F,φ ifn this is equiv to

$$F(a)\varphi(a^2)\varphi(a^3)\cdots\varphi(a^{m-1}) \,/\, \varphi(1)\varphi(a)\varphi(a^2)\cdots\varphi(a^{m-1})$$

where denom is sym. fn of roots of $x^m = 1$ and is numerical so it becomes an ifn ∴ by part 1) it is reducible.

3) If φ qfn we can reduce to ifn. ■

Note that the sum of the roots or their similar powers is always zero when index !divby m and sum equals m if index divby m. Also, take the time to grasp the significance of a fn being a sym. fn of the roots of an eqn. This idea will be built upon.

Prop. 7.24. a,b,c,… prime ⇒ roots of $x^{a+b+c+\cdots} = 1$ are terms of

$$(1+\alpha+\alpha^2+\cdots+\alpha^{a-1})(1+\beta+\beta^2+\cdots+\beta^{b-1})(1+\gamma+\gamma^2+\cdots+\gamma^{c-1})\cdots$$

where α is a root of $x^a = 1$ and so forth.

Cor. 1. This holds if a,b,c,… are powers of prime numbers.

If p(p,a) we know that $a^{p-1} - 1$ is divby p from Fermat. Further (q < p-1) ∧ (a^p- 1 divby p) ⇒ p - 1 divby q. If a causes no such numbers as q to exist then a is a **primitive root of p**.

p	**a≡prim.root**
3	2
5	2,3
7	3,5
11	2,6,7,8
and so on	

Thm. 7.1. a ≡ primitive root of p ⇒ dividing a, a^2, a^3, …, a^{p-1} by p leaves different remainders.

Proof

Else a^m, $a^{m'}$ have same remainder (m > m')

∴ $a^m - a^{m'}$ divby p

∴ $a^{m-m'} - 1$ divby p and p($a^{m'}$,p) ↴ because a is not some remainder≡primitive root ■

Prop. 7.25. α root of $x^p = 1$, α ∈ **C-R**, p prime, α primitive root of p ⇒ all roots are

$$1, \alpha^{\alpha}, \alpha^{\alpha\wedge 2}, \alpha^{a\wedge 3}, \ldots \alpha^{\alpha\wedge(p-1)}$$

Proof

Let α^m be the index of any element in this series.

Divide α^m by p. Let this quotient be c, remainder r.

∴ $\alpha^m = cp + r$

∴ $\alpha^{\alpha\wedge m} = (\alpha^p)^c \cdot \alpha^r = \alpha^r$

By above r ∈ {1,2,3,…,(p-1)} ∴ series reordered is 1, α, α^2, …, α^{p-1}

∴ These are all the roots. ■ Note that here $\alpha^{p-1} = \alpha$.

With this form of prime numbers nailed down, Euler, Lagrange, and Gauss pursued these ideas into functions. To follow them, we will actually need this next weird thing. From $x^p = 1$, we have its roots α, α^α, …, $\alpha^{\alpha\wedge(p-2)}$ as above and from $y^{p-1} = 1$, we have its roots 1, ω, ω^2, …, ω^{p-2} where these are its roots of unity. And (weirdly, but you'll see why in a minute) we want the sum of these terms' products:

$$V = 1\cdot\alpha + \omega\cdot\alpha^{\alpha} + \omega^2\cdot\alpha^{\alpha\wedge 2} + \cdots + \omega^{p-2}\alpha^{\alpha\wedge p-2}$$

Now V^{p-1} is a fn of α, α^α, …, which does not change if we swap α^α for α (or any similar adjacent swap -- think substitution group). Sub α^α for α and V becomes V_1. Then α = $\omega^{p-1}\alpha^{\alpha\wedge p-1}$ ∴ $V^{p-1} = V_1^{p-1}$. So for any swap of adjacent terms we get this same kind of result where $V^{p-1} = V_i^{p-1}$. And if ω should have a power greater than p-2, we can depress it as $\omega^{p-1} = 1$, $\omega^p = \omega$, and so on. So if V^{p-1} has form $A_1 + \omega A_2 + \omega^2 A_3 + \cdots + \omega^{p-2} A_{p-1}$ then because V^{p-1} never changes, neither do the A_i.

If we express A_i in terms of α^m:

$$A_1 = a_1 + b_1\alpha + c_1\alpha^{\alpha} + \cdots + p_1\alpha^{\alpha\wedge(p-2)}$$

Sub α^α for α

$$A_1 = a_1 + b_1\alpha^{\alpha} + c_1\alpha^{\alpha\wedge 2} + \cdots + p_1\alpha^{\alpha\wedge(p-1)}$$

∴ $b_1 = c_1$, $c_1 = d_1$, … $p_1 = b_1$

∴ $A_1 = a_1 + b_1(\alpha + \alpha^{\alpha} + \alpha^{\alpha\wedge 2} + \cdots + \alpha^{\alpha\wedge(p-2)})$

Here a_1 is independent of α and b_1 is any coeff we choose. So these are known. And $1+\alpha+\alpha^2+\cdots+\alpha^{p-1} = 0$ ∴ $A_1 = a_1 - b_1$ $A_2 = a_2 - b_2$ $A_3 = \ldots$. So V^{p-1} is completely known and we can derive V_1. Because p-1 is not prime, we can use its factors to reduce the roots of unity to get V_1. Let's show this V stuff in use. For some prime p, we want all the roots of $x^p = 1$. Then

$$1, \omega_1, \omega_2, \ldots, \omega_{p-2} \text{ are the roots of } y^{p-1} = 1$$
$$V_0 = \alpha + \alpha^{\alpha} + \alpha^{\alpha\wedge 2} + \cdots + \alpha^{\alpha\wedge p-2}$$
$$V_1 = \alpha + \omega_1\alpha^{\alpha} + \omega_1^{2}\alpha^{\alpha\wedge 2} + \cdots + \omega_1^{p-2}\alpha^{\alpha\wedge p-2}$$
$$V_2 = \alpha + \omega_2\alpha^{\alpha} + \omega_2^{2}\alpha^{\alpha\wedge 2} + \cdots + \omega_2^{p-2}\alpha^{\alpha\wedge p-2}$$
$$\cdots$$
$$V_{p-1} = \alpha + \omega_{p-2}\alpha^{\alpha} + \omega_{p-2}^{2}\alpha^{\alpha\wedge 2} + \cdots + \omega_{p-2}^{p-2}\alpha^{\alpha\wedge p-2}$$

$V_0 = -1$. V_2 and V_3 are derived from V_1 by subbing ω_2 or ω_3 for ω_1. And calculating V_1 as above determines the others.

∴
$$V_0 + V_1 + V_2 + \cdots + V_{p-2} = (p-1)\alpha$$
$$V_0 + \omega_1^{p-1}V_1 + \omega_1^{p-1}V_2 + \cdots + \omega_1^{p-1}V_{p-2} = (p-1)\alpha^{\alpha}$$
$$V_0 + \omega_1^{p-2}V_1 + \omega_1^{p-2}V_2 + \cdots + \omega_1^{p-2}V_{p-2} = (p-1)\alpha^{\alpha\wedge 2}$$
$$\cdots$$

All of this leads to calculating the numerical value of roots. Applying it to $x^5 = 1$:

$$4\alpha = -1 + \sqrt{5} + \sqrt[4]{(-15+20i)} + \sqrt[4]{(-15-20i)}$$

The rational part of the roots of an eqn is the coeff of the second term divided by the degree. Here $(x^5 - 1)/(x - 1)$ has roots $\alpha, \alpha^2, \alpha^3, \alpha^4$ and for

$$x^4 + x^3 + x^2 + x + 1 = 0$$

the rational part of the roots is -1/4. We can simplify the above, thanks to Lagrange and we will go into his work in order to pursue this a little further. Because p is prime, p-1 is compound. Let p-1 equal c·b with c,b primes. Let ω≠1 be root of $z^c = 1$ then ω is a root of $y^{p-1}= 1$. Then V is composed of these next b terms where powers of ω occur in the same order:

$$U_0 = \alpha + \alpha^{\alpha\wedge c} + \alpha^{\alpha\wedge 2c} + \cdots + \alpha^{\alpha\wedge(b-1)c+1}$$
$$U_1 = \alpha^{\alpha} + \alpha^{\alpha\wedge c+1} + \alpha^{\alpha\wedge 2c+1} + \cdots + \alpha^{\alpha\wedge(b-1)c+2}$$
$$U_2 = \alpha^{\alpha\wedge 2} + \alpha^{\alpha\wedge c+2} + \alpha^{\alpha\wedge 2c+2} + \cdots + \alpha^{\alpha\wedge(b-1)c+2}$$
$$\cdots$$
$$U_{c-1} = \alpha + \alpha^{\alpha\wedge c-1} + \alpha^{\alpha\wedge 2c-1} + \cdots + \alpha^{\alpha\wedge p-2}$$

In this, V becomes $U_0 + \omega U_1 + \omega^2 U_2 + \cdots + \omega^{c-1}U_{c-1}$ and V^c becomes our unchanging fn. Let ω' be a root of $u^b = 1$ then we have

$$U = \alpha + \omega'\alpha^{\alpha\wedge c} + \omega'^{2}\alpha^{\alpha\wedge 2c} + \cdots + \omega'^{b-1}\alpha^{\alpha\wedge(b-1)c}$$

and u^b is like our earlier V^{p-1} for getting values of α, $\alpha^{\alpha\wedge c}$, and so on.

Example

$x^\mu = 1 \equiv x^5 - 1 = 0$
μ-1 = 4 ∴ c = 2, b = 2 ω,ω' roots of $y^2 = 1$ (also $y^4 = 1$)
The primitive root for 5 is 2 (you can always use the first one if there are more)
α is a root of $x^5 = 1$.
$\therefore\ V = \alpha + \omega\alpha^2 + \omega^2\alpha^{2\wedge 2} + \omega^3\alpha^{2\wedge 3}$
$\quad = (\alpha + \alpha^4) + \omega(\alpha^2 + \alpha^3)$
$\quad = U_0 + \omega U_1$
$\therefore V^2 = (U_0{}^2 + U_1{}^2) + 2\omega U_1 U_2$
$\quad U_0 = \alpha + \alpha^4 \qquad U_0{}^2 = \alpha^2 + \alpha^3 + 2$
$\quad U_1 = \alpha^2 + \alpha^3 \qquad U_1{}^2 = \alpha^4 + \alpha + 2$
$\therefore U_0{}^2 + U_1{}^2 = 4 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 3$
$\quad U_0 U_1 = \alpha^3 + \alpha + \alpha^4 + \alpha^2 = -1$
$\therefore V^2 = 3 - 2\omega$
Let ω = -1
$\therefore V_0 = \alpha + \alpha^2 + \alpha^{2\wedge 2} + \alpha^{2\wedge 3} = -1 = U_0 + U_1$
$\quad V_1 = \alpha + \omega\alpha^2 + \omega^2\alpha^{2\wedge 2} + \omega^3\alpha^{2\wedge 3} = \sqrt{5} = U_0 - U_1$
$\therefore 2U_0 = -1 + \sqrt{5} \qquad 2U_1 = -1 - \sqrt{5}$
Now we do the $U = \alpha + \omega'\alpha^{2\wedge 2}$ where ω' = ω = -1
$U^2 = (\alpha + \omega'\alpha^4)^2 = \alpha^2 + \omega'^2\alpha^3 + 2\omega'$
$\alpha^2 + \alpha^3 = U_1 = ½(-1 - \sqrt{5}) \qquad 2\omega' = -2$
$\therefore U^2 = ½(-5 \sqrt{-}\ 5)\ \therefore\ \alpha - \alpha^4 = \sqrt{(½(-5 - \sqrt{5}))} \quad \alpha + \alpha^4 = ½(-1 + \sqrt{5})$

That's the sort of thing Lagrange developed this for. Again, as with V^{p-1} etc., we have an algebraic method for determining the numerical value of roots for $x^5 = 1$. This idea is also used by Lagrange to find the roots of more general eqns. Here we are looking at the form of number as Galois found it. Let x_i [1-n] be root of an n° eqn and ω be a root of $y^n = 1$ and

$$V = x_1 + \omega x_2 + \omega^2 x_3 + \cdots + \omega^{n-1} x_n$$

If we permute the roots -- in our earlier notation, $(x_1,x_2,x_3,\ldots,x_n)$ -- as a substitution group, we have

$$V^n = X_0 + \omega X_1 + \omega^2 X_2 + \cdots + \omega^{n-1} X_{n-1}$$

where the X_i do not change under the substitutions. The values of V^n under these substitutions are

$$V_0 = x_1 + x_2 + x_3 + \cdots + x_n$$
$$V_1 = x_1 + \omega x_2 + \omega^2 x_3 + \cdots + \omega^{n-1} x_n$$
$$V_2 = x_1 + \omega^2 x_2 + \omega^4 x_3 + \cdots + \omega^{2n-2} x_n$$
$$\cdots$$
$$V_{n-1} = x_1 + \omega^{n-1} x_2 + \omega^{n-2} x_3 + \cdots + \omega x_n$$

V_0 is our fn's second coeff with the sign changed. If we add these, after multiplying each by the power of the roots of unity supplementary to n regarding the root we are going after, we have

$$nx_1 = V_0 + V_1 + V_2 + \cdots + V_{n-1}$$
$$nx_2 = V_0 + \omega^{n-1}V_1 + \omega^{n-1}V_2 + \cdots + \omega V_{n-1}$$
$$nx_3 = V_0 + \omega^{n-2}V_1 + \omega^{n-4}V_2 + \cdots + \omega^2 V_{n-1}$$
$$\cdots$$
$$nx_n = V_0 + \omega V_1 + \omega^2 V_2 + \cdots + \omega^{n-1}V_{n-1}$$

If our eqn is $x^{n-1} - ax^{n-2} + bx^{n-3} - \cdots = 0$ then

$$a = V_1^n + V_2^n + \cdots + V_{n-1}^n$$
$$b = V_1^nV_2^n + V_1^nV_3^n + \cdots + V_{n-2}^nV_{n-1}^n$$
$$\cdots$$

Example

$x^3 - Ax^2 + Bx + C$ roots x_1, x_2, x_3
$V = x_1 + \omega x_2 + \omega^2 x_3$
$V^3 = X_0 + \omega X_1 + \omega^2 X_2$
$X_0 = x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3$
$X_1 = 3(x_1^2x_2 + x_2^2x_3 + x_3^2x_1)$
$X_2 = 3(x_2^2x_1 + x_3^2x_2 + x_1^2x_3)$
X_1, X_2 are coeffs of a quadratic where $X_1 + X_2$ and X_1X_2 are sym. fns of its roots
$\therefore\ V_0 = x_1 + x_3$
$V_1 = x_1 + \omega x_2 + \omega^2 x_3$
$V_2 = x_1 + \omega^2 x_2 + \omega x_3$
These give the roots as V_1^3 and V_2^3 and are what V becomes when for unity we sub $\frac{1}{2}(-1\pm\sqrt{-3})$.

The next method, originating with Bezout, is essentially the above with binomials and **usually fails** with polynomials which are not bi. But let's view it as a glimpse of the clockwork underlying eqns. We want the roots of an n° φx with coeff of $x^n = 1$. An f(y) is any n° eqn in y with roots y_i [1-n] which are known quantities. Let x = F(y) where F has n coeffs so that φx is the result in eliminating y in f(y) = 0 and x - F(y) = 0. So our F(y) is (n-1)°. We have

$(x - F(y_1))(x - F(y_2))(x - F(y_3))\cdots(x - F(y_n)) = \varphi x$
$\therefore \ln(1 - F(y_1)/x) + \ln(1 - F(y_2)/x) + \cdots \ln(1 - F(y_n)/x) = \ln(\varphi x/x^n)$
Let $\varphi x/x^n = A_1/x + A_2/x^2 + A_3/x^3 + \cdots$ and
$F(y_1) + F(y_2) + \cdots + F(y_n) = S_1$
$(F(y_1))^2 + (F(y_2))^2 + \cdots + (F(y_n))^2 = S_2$
$\cdots$
$\therefore -(S_1/x + \frac{1}{2}S_2/x^2 + \frac{1}{3}S_3/x^3 + \cdots) = A_1/x + A_2/x^2 + A_3/x^3$
And doing our coeffs of equal series thing: $S_1 = A_1$ $S_2 = -2A_2$ $S_3 = -3A_3$ ... $S_n = -nA_n$
These are n eqns between n unknown coeff of F(y). Determine(good luck) these coeffs and our roots are $x_1 = F(y_1), x_2 = F(y_2), \cdots, x_n = F(y_n)$

Example

$\varphi x = x^3 + ax + b = 0$
$f(y) = y^3 - 1 = 0$ roots $1, \omega, \omega^2$
$x = qy + ry^2$
$\therefore (x - (q+r))(x - (q\omega + r\omega^2))(x - (q\omega^2 + r\omega)) = x^3 + ax + b$
$\therefore \ln(1 - (q+r)/x) + \ln(1 - (q\omega + r\omega^2)/x)(1 - (q\omega^2 + r\omega)/x)$
$= \ln(1 + (a/x^2 + b/x^3)) = a/x^2 + b/x^3 - ½a^2/x^4 \cdots$
$\therefore (q + r)^2 + (q\omega + r\omega^2)^2 + (q\omega^2 + r\omega)^2 = -2a$
$(q + r)^3 + (q\omega + r\omega^2)^3 + (q\omega^2 + r\omega)^3 = -3b$
Then as $1 + \omega^2 + \omega^4 = 1$, $1 + \omega^3 + \omega^6 = 3$, these are
$3qr = -a$
$q^3 + r^3 = -b$
and we solve for q and r.

More Solutions of Equations

With Murphy, we are on the cusp of Galois Theory, which we will not deal with in this text beyond this next "long story (way too) short": For, say, a 4° eqn, we have the perm of the roots $(x_1x_2x_3x_4)$. If this forms the group G_4, which can be reduced down through G_3 and G_2 to G_1, we can solve by radicals. This is always possible for degrees up through four. It breaks down on quintics and all the old attempts for a general soln had to fail. Being human, we need another, even more radical solution than our by radicals, approach. So let's look at a couple of approaches which were found to sometimes work to see what we can see -- the other side of the mountain, if nothing else.

Thm. 7.2. Denote $x(x - h)(x - 2h)\cdots(x - (n-1)h)$ as $[x]^n$. These n factors are in A.P. ⇒ $[x+y]^n = [x]^n + n[x]^{n-1}[y] + n(n-1)/2!\ [x]^{n-2}[y]^2 + n(n-1)(n-2)/3!\ [x]^{n-3}\ [y]^3 + \cdots$
where the special case of $h = 0$ gives us the Binomial Theorem.

I leave the proof to your curiosity and you **should** be curious. Let's see this in use.

Example

$[x]^2 + ax = b$ or $x(x-h) + ax = b$
add $[a/2]^2$ to each side
$\therefore [x]^2 + ax + [a/2]^2 = b + [a/2]^2$
$\therefore [x + a/2]^2 = b + [a/2]^2$ where LHT $\equiv (x + a/2)(x + a/2 - h)$

...such in this case is the simple eqn to which the proposed quadratic is reducible. Thus it will be seen that the soln of eqns in the algebraic sense consists in reducing them to binomials of a particular form and that form has the advantage which contains only pure powers of the unknown quantity; but the question admits of extension to any form of fn in which x may be regularly involved.

Remember that. Murphy, like Chrystal, also solves eqns by series. Let's look at his method and his first and simplest example.

Thm. 7.3. Take φx, arrange in ascending powers of x and make the coeff of x^1 equal unity. Then take the ln of the fn. Take the coeff of 1/x in each term and this infinite sum with sign changed is a root of the fn.

Proof

$\varphi x = A(x - \alpha)(x - \beta)(x - \gamma)\cdots$ where A is independent of x

$= A'(x - \alpha)(1 - x/\beta)(1 - x/\gamma)\cdots$ where $A' = A(-\beta)(-\gamma)\cdots$

$\therefore\ \varphi x/x = A'(1 - \alpha/x)(1 - \beta/x)(1 - \gamma/x)\cdots$

$\ln \varphi x/x = \ln A' + \ln(1 - \alpha/x) + \ln(1 - \beta/x) + \cdots$

$\therefore \ln(1 - \alpha/x) = -\alpha/x - ½\, \alpha^2/x^2 - ⅓\, \alpha^3/x^3 + \cdots$

And the coeff of 1/x is -α which is the root with the sign changed ■

Example

Given $x^2 + ax + b$ find root α

$\varphi x/x = a + x + b/x$

$\ln(\varphi x/x) = \ln a + \ln(1 + z)$ where $z = 1/a(x + b/x)$

$\therefore -\alpha =$ coeff of 1/x in $z - z^2/2 + z^3/3 + z^4/4 + \cdots$

coeff of 1/x in $z = b/a$

in $z^3 = 3b^2/a^3$

in $x^5 = 10b^3/a^5$

and so on and 1/x does not appear in even powers of z

$\therefore \alpha = -(b/a + b^2/a^3 + 2b^3/a^5 + 5b^4/a^7 + \cdots)$

The general term $z^{2n+1}/2^{2n+1}$ and its coeff of 1/x is

$$(2n-1)(2n-2)(2n-3)\cdots(n+2)/n! \cdot b^{n+1}/a^{2n+1}$$

$\therefore \alpha = -(b/a + b^2/a^3 + 4/2\ b^3/a^5 + 6{\cdot}5/2{\cdot}3\ b^4/a^7 + \cdots)$

If we consider $x = -a/2 + \sqrt{(a^2/4 - b)}$, the soln by radicals,

$= -a/2 + a/2(1 - 1b/a^2)^{1/2}$

$= a/2(1/2\ 4b/a^2 + 1{\cdot}1/2{\cdot}4\ (4b/a^2)^2 + 1{\cdot}1{\cdot}3/2{\cdot}4{\cdot}6\ (4b/a^2)^3 + \cdots)$

Let S_n be the nth term of either of these series. Then

$$S_n = (2n-1)(2n-2)(2n-3)\cdots(n+2)/n! \cdot b^{n+1}/a^{2n+1}$$

$$S_{n-1} = (2n-2)(2n-3)(2n-4)\cdots(n+1)/n! \cdot b^{n}/a^{2n-1}$$

$$\therefore\quad S_n = 2n(2n-1)/(n+1)n \cdot b/a^2 \cdot S_{n-1}$$

$$= (1 - 1/2n)/(1 + 1/n) \cdot 4b/a^2 \cdot S_{n-1}$$

$n \to \infty$, if $4b > a^2$, $S_n > S_{n-1}$ and series diverges ∴ i-roots

Let's look at the roots α,β (α < β) when series converges:

$$\alpha + \beta = -a \qquad \alpha\beta = b$$

$$\therefore\quad \text{root} = \alpha\beta/\alpha+\beta + \alpha^2\beta^2/(\alpha+b)^3 + 2\alpha^3\beta^3/(\alpha+\beta)^5 + \cdots$$

$$\therefore\quad \alpha\beta/(\beta+\alpha) = \alpha\beta(\beta+\alpha)^{-1} = \alpha(1 - \alpha/\beta + \alpha^2/\beta^2 + \alpha^3/\beta^3 + \cdots)$$

and this series converges.

End of Example

Murphy uses all of these last few ideas to find an inverse fn of φx.
Let $\varphi(h) = x \Rightarrow h = \varphi^{-1}(x)$
$\therefore\ h = (x - \varphi(0)) \cdot h/(\varphi(h) - \varphi(0))$
Let $x - \varphi x = \xi$ and $x(\varphi(x) - \varphi(0)) = f(x)\ \therefore h = \xi f(h)$
$= \xi f(h) + \xi^2/2!\ (f(h)^2)' + \xi^3/3!\ (f(h)^3)'' + \cdots$
h is assumed to be zero. Therefore:

$$\varphi^{-1}(x) = (x\text{-}\varphi 0)[x/(\varphi x\text{-}\varphi 0)] + (x\text{-}\varphi 0)^2/2![(x/(\varphi x\text{-}\varphi 0)^2]' + (x\text{-}\varphi 0)^3/3![(x/(\varphi x\text{-}\varphi 0)^3]'' \cdots$$

where x = 0 in the square brackets. $x \to 0$, $\varphi x \to 0$ and $\varphi^{-1}(x) =$

$$x[x/\varphi x] + x^2/2![(x/\varphi x)^2]' + x^3/3![(x/\varphi x)^3]'' + \cdots$$

Example

$\varphi x = ax + bx^2 + cx^3 + dx^4 + \cdots$
$x/\varphi x = (a + bx + cx^2 + \cdots)^{-1}$ where const term is 1/a
$(x/\varphi x)^2 = (a + bx + cx^2 + \cdots)^{-2}$ where coeff x is $-2ba^{-3}$
$(x/\varphi x)^3 = (a + bx + cx^2 + \cdots)^{-3}$ where coeff x^2 is $1/2!\ [(x/\varphi x)^3]'' = -3ca^{-4} + 6b^2a^{-5}$
and so on $\therefore\ \varphi^{-1}(x) = x/a - bx^2/a^3 + (2b^2\text{-}ac)x^3/a^5 + \cdots$

Recurring Series

Murphy, and these mathematicians he draws upon, makes great use of recurring series. A bit of this will be a review of Chrystal. But not much.

$$S = u_1 + u_2 + u_3 + \cdots + u_x + \cdots$$

where u_x is the general term and is the sum of some number of preceding terms multiplied by a constant. If m is the number of constants, then we need m arbitrary, but often **well-chosen**, terms to build the recurring series or 2m quantities define a recurring series. The simplest recurring series are in G.P.

Examples

1) One constant 3 and one term 2
S = 2, 6, 18, 54, 162, …

2) Two constants 1,3 and two terms 2,4
S = 2, 4, 2·1+4·3, 4·1+14·3, 14·1+46·3, …
= 2, 4, 14, 46, 152, …

3) Three constants -1,0,1 and three terms 1,2,3
S = 1, 2, 3, -1·1+0·2+1·3, …
= 1, 2, 3, 2, 0, -3, -5, -5, -2, 3, 8, 10, … (We'll see this one again.)

The sum of two recurring G.P. series is a recurring series with two constants of relation. So these are u_x with constant α and v_x with constant β. ∴

$w_x = u_x + v_x$
We eliminate x in $u_{x+1} = \alpha u_x$, $v_{x+1} = \beta v_x$, $w = u_x+v_x$ ∴
$w_x = u_x + v_x$ [1]
$w_{x+1} = u_{x+1}+v_{x+1} = \alpha u_x+\beta v_x$ [2]
$w_{x+2} = u_{x+2}+v_{x+2} = = \alpha^2 u_x+ \beta^2 v_x$ [3]
Multiply [1] by λ' and [2] by λ" and equate the sum of these to [3]
∴ $w_{x+2} = \lambda' w_x + \lambda'' w_{x+1}$ where $\alpha^2 = \lambda' + \lambda''\alpha$ and $\beta^2 = \lambda' + \lambda''\beta$
∴ α,β are roots of $z^2 = \lambda' + \lambda'' z$
∴ $\lambda' = -\alpha\beta$ and $\lambda'' = \alpha+\beta$
∴ $w_{x+2} = -\alpha\beta w_x + (\alpha+\beta) w_{x+1}$
So we have two constants -αβ and (α+β) and two terms $u_1+v_1=w_1$ and $\alpha u_1+\beta v_1 = w_2$.

Example

u_i = 3, 6, 12, 24, 48, 96, … α = 2
v_i = 1, 3, 9, 27, 81, 243, … β = 3
$w_i = u_i + v_i$ = 4, 9, 21, 51, 129, 339, …
where constants are -2·3 = -6 and (2+3) = 5
21 = -6·4 + 5·9 51 = -6·9 + 5·21 and so on.

Sym. with 3 G.P. series, constants α,β,γ get us $w_{x+3} = \lambda' w_x + \lambda'' w_{x+1} + \lambda''' x_{x+2}$ where $\lambda' = \alpha\beta\gamma$, $\lambda'' = (\alpha\beta + \beta\gamma + \gamma\alpha)$, $\lambda''' = \alpha + \beta + \gamma$. In general, with n G.P. series, we can build a Sym. proof in this form. Let a', a", a'", … be the constants, u', u", u'", … be the series. Then summing to series w:

$$w_x = u'_x + u''_x + u'''_x + \cdots$$
$$w_{x+1} = a'u'_x + a''u''_x + a'''u'''_x + \cdots$$
$$w_{x+2} = a'^2u'_x + a''^2u''_x + a'''^2u'''_x + \cdots$$
$$\cdots$$
$$w_{x+n} = a'^n u'_x + a''^n u''_x + a'''^n u'''_x + \cdots$$

Then take n arbitrary constants λ', λ", λ'", …, use them as above multiplying each line i by $\lambda^{i\prime}$, sum the lines, equate sum to n+1 and

$$w_{x+n} = \lambda' w_x + \lambda'' w_{x+1} + \cdots + \lambda_{(n)} w_{x+n-1}$$

and we find that a',a",a'",… are roots of

$$z^n = \lambda' + \lambda'' z + \lambda''' z^2 + \cdots + \lambda^{(n)} z^{n-1}$$

and the λ^i are given us by the relations of roots to coeffs.

So the G.P. of the sum of n G.P.s ha n a^i constants and n terms:

$$w_1 = u'_1 + u''_2 + \cdots$$
$$w_2 = a'u'_1 + a''u''_2 + \cdots$$
$$\cdots$$
$$w_n = a'^{n+1}u'_1 + a''^{n+1}u''_2 + \cdots$$

If m constants are the same, then there are n-m+1 constants in this result. Now here is why we even care about all this. If you have a recurring series

$$u_1 + u_2 + u_3 + \cdots$$

and multiply it term by term with a given series

$$1 + z + z^2 + \cdots$$

you get

$$u_1 + u_2z + u_3z^2 + \cdots \qquad [1]$$

and now we have a polynomial which is subject to these new tools. This [1] is the expansion of an ifrac

$$\frac{a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}}{1 - \lambda'z^n - \lambda''z^{n-1} - \cdots - \lambda^{(n)}z}$$

The coeffs of [1] are a recurring series

$$a_0 = u_1$$
$$a_1 = u_2 - \lambda^{(n)}u_1$$
$$a_2 = u_3 - \lambda^{(n-1)}u_1 - \lambda^{(n)}u_2$$
$$a_3 = u_4 - \lambda^{(n-2)}u_1 - \lambda^{(n-1)}u_2 - \lambda^{(n)}u_3$$
$$\cdots$$
$$a_{n-1} = u_n - \lambda''u_1 - \lambda'''u_2 - \cdots - \lambda^{(n)}u_{n-1}$$

And we can turn this around and find the ifrac if given the polynomial or even just the first bit of the polynomial.

Examples

1) Find the ifrac of $2 + 4z + 14z^2 + 46z^3 + 152z^4 + \cdots$

$$\lambda' = 1 \quad \lambda'' = 3 \quad u_1 = 2 \quad u_2 = 4$$
$$a_0 = 2 \quad a_1 = 4 - 6 = -2$$
$$\therefore (2 - 2x)/(1 - z^2 - 3z)$$

2) And for $1 + 2x + 3x^2 + 2x^3 - 3x^5 - 5x^6 - \cdots$

$u_1 = 1 \quad u_2 = 3 \quad u_3 = 3 \quad \lambda' = -1 \quad \lambda'' = 0 \quad \lambda''' = 1$
$a_0 = 1 \quad a_1 = 2 - 1 = 1 \quad a_3 = 3 - 1 \cdot 2 = 1$
$\therefore (1 + z + z^2)/(1 + z^3 - z)$

And this leads us back to partial fractions. With our root eqn:

$$\varphi(z) = C(z - \alpha')(z - \alpha'')\cdots(z - \alpha^{(n)})$$

where α^i are the n roots and C is independent of z. Let f(z) be

$$A_0 + A_1 z + A_2 z^2 + \cdots + A_{n-1} z^{n-1}$$

with n constants. Suppose we use a common denom to add

$$A'/(z - \alpha') + A''/(z - \alpha'') + \cdots + A^{(n)}/(z - \alpha^{(n)})$$

where the A primes are not the A_i's in f. The common denom becomes $1/C \cdot \varphi(z)$ and the num is

$$A'(z-\alpha'')(z-a''')\cdots + A''(z-\alpha')(z-\alpha''')\cdots + \cdots$$

and this is f(z), if we use n eqns to determine the A primes:

$A' + A'' + \cdots + A^{(n)} = A_{n-1}$
$A'(\alpha''+\alpha'''+\cdots) + A''(\alpha'+\alpha'''+ \cdots) + \cdots = -A_{n-2}$
$A'(\alpha''\alpha''' + \cdots) + A''(\alpha'\alpha''' + \cdots) + \cdots = A_{n-3}$ (These are sums $C_{n|2}$)
$\cdots$

We know about equal roots here. Let's ignore them for now. And let's assume C = 1. Then

$$f(z)/\varphi(z) = A'/(z-\alpha') + A''/(z-\alpha'') + \cdots + A^{(n)}/(z-\alpha^{(n)})$$

$\therefore f(z)/(z-\alpha'')(z-\alpha''')\cdots =$

$$A' + A''(z-\alpha')/(z-\alpha'') + A'''(z-\alpha')/(z-\alpha''') + \cdots$$

and if we make $z = \alpha'$

$f(\alpha')/(\alpha'-\alpha'')(\alpha'-\alpha''')\cdots(\alpha'-\alpha^{(n)}) = A'$
$\therefore \quad f(\alpha')/\varphi'(\alpha') = A'$

And Sym. $A'' = f(\alpha'')/\varphi'(\alpha'')$ $A''' = f(\alpha''')/\varphi'(\alpha''')$ and so on. So we can easily calculate our fractions and if $C \neq 1$, we can stick it back in. You may have picked all this up the first time. But I'm repeating Chrystal, via Murphy, to emphasize the part played by φ'.

Examples

1) Decompose $1/(z-\alpha)(z-\beta)$

$$f(z) = 1 \quad \varphi(z) = z^2 - (\alpha+\beta)z + \alpha\beta \quad \varphi'(z) = 2z - (\alpha+\beta)$$
$$f(z)/\varphi'(z) = 1/(2z - (\alpha+\beta))$$

Put α,β successively for z and we get the nums of our fractions: $1/(\alpha-\beta)$ $1/(\beta-\alpha)$

$$\therefore \quad 1/(z-\alpha)(z-\beta) = 1/(\alpha-\beta) \cdot (1/(z-\alpha) - 1/(z-b))$$

2) Decompose $(2z + 1)/z(z+1)(z+2)$

Sub roots of the denom $(0, -1, -2)$ into $f(z)/\varphi'(z)$ and the nums are $1/2, 1, -3/2$

$$\therefore \quad ½(1/z + 2/z+1 - 3/z+2)$$

If $f(z) = \varphi'(z)$ all the nums are unity:

$$\varphi'(z)/\varphi(z) = 1/z-\alpha' + 1/z-\alpha'' + 1/z-\alpha''' + \cdots$$

Let's do a long example to see where this can lead. We'll decompose

$$1/(\varepsilon^{hz} - 1) = \varepsilon^{-h/2^z}/(\varepsilon^{h/2^z}-\varepsilon^{-h/2^z} = ½ ((\varepsilon^{h/2^z}+ \varepsilon^{-h/2^z})/(\varepsilon^{h/2^z}- \varepsilon^{-h/2^z})) - ½$$

By what we know of trig, if we make that last denom equal to zero, then for $\forall m\in\mathbf{Z}$,

$$z = (2m\pi i)/h$$

which gives z infinite values. Now make that denom our $\varphi(z)$ and

$$\varphi(z) = 2(h/2\cdot z + (h/2)^3\cdot z^3/3! + (h/2)^5\cdot z^5/5! + \cdots)$$
$$\therefore \quad \varphi'(z) = h(1 + (h/2)^2\cdot z^2/2! + (h/2)^4\cdot z^4/4! + \cdots)$$
$$= h/2 \cdot (\text{ our numerator })$$

$$\therefore \ 1/(\varepsilon^{hz} - 1) = 1/h \cdot \varphi'(z)/\varphi(z) - 1/2$$

$$= 1/2 + 1/h(1/z + (z - 2\pi i/h)^{-1}+ (z - 4\pi i/h)^{-1} + \cdots + (z + 2\pi i/h)^{-1}+ (z + 4\pi i/h)^{-1}+ \cdots$$

If we take these terms and expand them in pairs as in

$$(z - 2\pi i/h)^{-1} + (z + 2\pi i/h)^{-1} = 2h(1/2^2\pi^2\cdot hz - 1/2^4\pi^4\cdot h^3z^3 + \cdots)$$

then

$$1/(e^{hz}-1) = 1/hz - 1/2 + 2hz/\pi^2 (1/2^2 + 1/4^2 + 1/6^2 + \cdots)$$
$$+ 2h^3z^3/\pi^4 (1/2^4 + 1/4^4 + 1/6^4 + \cdots) + \cdots$$

Warning: The B_i which follow are **not** Bernoulli numbers.

If

$$1/1^2 + 1/2^2 + 1/3^2 + \cdots = 2B_1\pi^2$$
$$1/1^4 + 1/2^4 + 1/3^4 + \cdots = 2^3B_3\pi^4/3!$$
$$1/1^6 + 1/2^6 + 1/3^6 + \cdots = 2^5B_5\pi^6/5!$$

then

$$1/(\varepsilon^{hz}-1) = 1/h^2 - 1/2 + B_1hz + B_3h^3z^3/5! + B_5h^5z^5/5! + \cdots$$

To calculate these B_i:

$$\varphi'(z)/\varphi(z) = 1/(\varepsilon^z -1) = 1/2 = 1/z + B_1z + B_3z^3/3! + \cdots$$
$$\ln \varphi z = \ln z + \ln (1 + (1/2^2 \cdot z^2/3! + 1/2^4 \cdot z^4/5! + \cdots))$$

This logarithm is then

$$A_1z^2 - A_3z^4 + A_5z^6 - \cdots$$

which gives us

$$\varphi'z/\varphi z = 1/z + 2A_1z - 4A_3z^3 + 6A_5z^5 - \cdots$$

And using our equal series's coeff thingie, you should get:

$B_1 = 1/2^2\cdot 3$ $\quad$ $B_3 = 1/2^3\cdot 3\cdot 5$ $\quad$ and so on.

This leads to a restatement of our $\varphi'z/\varphi z$:

$$F(x) = f(x) - 1/2hf'(x) + B_1h^2f''(x) - B_3h^4/3!\cdot f^{(4)}(x) + B_5h^6/5!\cdot f^{(6)}(x) - \cdots$$

Then $F(x+h) - F(x) = h\cdot f'(x)$ and our "long example" ends.

Examples

1) $F(x) = x^2 \;\therefore\; F(x) = x^2 - hx + h^2/6$
$\therefore\; F(x+h) - F(x) = (2x + h)h - h^2 = 2xh = h\cdot f'(x)$

And we can use this to sum a series:

$$F(x) = u_1 + u_2 + u_3 + \cdots + u_x + \cdots$$
$$\therefore \quad F(x+1) = u_1 + u_2 + \cdots + u_x + u_{x+1}$$
$$\therefore \quad F(x+1) - F(x) = u_{x+1} \equiv f'(x)$$

Then $(f(x) + C)' = f'(x)$ as we know from the Calculus we saw in DME.

2) $F(x) = 1^3 + 2^3 + 3^3 + \cdots + x^3$

$F(x+1) = 1^3 + 2^3 + 3^3 + \cdots + x^3 + (x+1)^3$

$f'(x) = (x + 1)^3$ (simply by subtraction)

$f''(x) = 3(x + 1)^2$

$f^{(4)}(x) = 6$

∴ $F(x) = C + 1/4\cdot(x+1)^4 - 1/2\cdot(x + 1)^3 + 1/4\cdot(x+1)^3$

To determine C, let x = 1, the F(1) is the first term.

∴ $1 = C + 1/4\cdot(2^4 - 2^4 + 2^2)$ ∴ $C = 0$

∴ $F(x) = (x+1)^2((x+1)^2 - 2(x+1) + 1) = ((x(x+1)/2)^2$

∴ $1^3 + 2^3 + 3^3 + \cdots + x^3 = (1 + 2 + 3 + \cdots + x)^2$

But you knew that. Let's now consider partial fractions in this way when the denom has equal roots. Our proper ifrac this time is:

$f(x)/((z - \alpha')^m(z - \alpha'')(z - \alpha''')\cdots(z - \alpha^{(n)}))$

$\varphi z = (z - \alpha')^m(z - \alpha'')(z - \alpha''')\cdots(z - \alpha^{(n)})$

∴ $1/\varphi z = 1/\varphi'\alpha'\cdot 1/(z - \alpha') + 1/\varphi'\alpha''\cdot 1/(z - \alpha'') + \cdots$

$f(z) = (f(z)\cdot f(\alpha')) + f(\alpha') = (f(z) - f(a'')) + f(\alpha'') = \cdots$

∴ $f(z)/\varphi z = 1/\varphi'\alpha'\cdot(f(z)-f(\alpha'))/(z-\alpha') + 1/\varphi'\alpha''\cdot(f(z) - f(\alpha''))/(z-\alpha'') + \cdots$
$+ 1/\varphi'\alpha'\cdot f(\alpha')/(z-\alpha') + 1/\varphi'\alpha''\cdot f(\alpha'')/(z-\alpha'') + \cdots$

If we sub α'+h for α', we are led to

$\varphi z/z-\alpha' \cdot (z - \alpha' - h) = \varphi z(1 - h/(z - \alpha'))^2$

Then f(z)/φz becomes $f(x)/\varphi z\cdot(1 - h/(z-\alpha'))^{-1}$ and we can collect coeffs of say h^{m-1} which coeff is

$f(z)/((z - \alpha')^{m-1}\varphi z)$

And this, after much expansion, is

$f\alpha'/\varphi'\alpha' \cdot 1/(z - \alpha')^m + (f\alpha'/\varphi'\alpha')' \cdot 1/(z - \alpha')^{m-1} + (f\alpha'/\varphi'\alpha')'' \cdot 1/2!(z - \alpha')^{m-2} + \cdots$
$+ f\alpha''/((\alpha''-\alpha')^{m-1}\varphi'\alpha'') \cdot 1/(z - \alpha'') + f\alpha'''/((\alpha'''-\alpha')^{m-1}\varphi'\alpha''') \cdot 1/(z - \alpha''') + \cdots$

which you should compare to our earlier version by Chrystal.

Example

Decompose $(2z^3 +7z^2 + 6z + 2)/(z^4 + 3z^3 + 2z^2)$ Roots of denom: 0, 0, -2, -1
To get the nums of fractions with denoms z+2, z+1, let z = -2, -1 in fraction
∴ nums: -1/2, 1
$\varphi z = (z + 1)(z - \alpha')(z + 2)$ where if $\alpha' = 0$ then $\varphi'(\alpha') = (\alpha' + 1)(\alpha' + 2)$
∴$f(\alpha')/\varphi'(\alpha') = (2 + 6\alpha' + 7\alpha'2 + 2\alpha'^3)/(2 + 3\alpha' + \alpha'^2) = 1 + 3/2\cdot\alpha' + 3/4\cdot\alpha'' + \cdots$
$\alpha' = 1 \Rightarrow f(\alpha')/\varphi'(\alpha') = 1, (f(\alpha')/\varphi'(\alpha')' = 3/2$
∴ partial fractions: $-1/2\cdot 1/(z+2)$, $1/(z+1)$, $1/z^2$, $3/2\cdot 1/z$

Murphy gives his own unique approach in the following:

Thm. 7.4. P/Q proper ifrac of z where $Q = (z - \alpha')^m Q_1$. Expand P/Q_1 in form

$$A_0 + A_1(z - \alpha') + A_2(z - \alpha')^2 + \cdots + A_{n-1}(z - \alpha')^{n-1}$$

Then $A_0/(z - \alpha')^n$, $A_1/(z - \alpha')^{n-1}$, ⋯, $A_{n-1}(z - \alpha')$ are partial fractions. Let $(z - \alpha'')^p$ be another factor of Q: $(z - \alpha'')^p Q_2 = Q$. Expand again in same form. Repeat for all factors of Q for all partial fractions.

Example

$$\text{Decompose } Z = (1/(z^3 - z))^2$$
$$Z = 1/z^2 \cdot (1 - z)^{-2} = 1/z^2 + 2/z + \cdots$$
$$Z = 1/(1 - z^2) \cdot (1 - (1 - z))^{-2} = 1/(1-z)^2 + 2/(1-z) + \cdots$$
$$\therefore \quad Z = 1/z^2 + 1/(1-z)^2 + 2/z + 1/(1-z)$$

Partial fractions of proper ifracs are all of the form

$$A/(a - z)^n$$

which expand as

$$A/a^n(1 + n \cdot z/a + n(n+1)/2! \cdot z^2/a^2 + \cdots)$$

and this is a **figurate** series. Its coeffs of z/a are figurate numbers. The mth figurate number of any order is the sum of m figurate numbers of the next inferior order. Here the coeff of z^m are

$$(1 - z)^{-(n+1)} = (1 - z)^{-1}(1 - z)^{-n}$$

Since the expansion of a proper ifrac by powers of z is a recurring series, every recurring series can be decomposed into a figurate series and if there are no equal roots in the denom then these series are geometrical. Let $Aa^n/(a - z)^n$ be a partial fraction. The coeff of z^x in the expansion is

$$A \cdot n(n+1)(n+2)\cdots(n+x-1)/x! \cdot (1/a)^x = A(x+1)(x+2)\cdots(x+n-1)/(n-1)! \cdot (1/a)^x$$

and collecting the coeff of x from each partial fraction, the sum is the coeff of x in the recurring series.

Solution by Recurring Series

Let A_i be the constants of relation in a recurring series with the general term u_x then

$$u_{x+n} = A_1u_x + A_2u_{x+1} + A_3u_{x+2} + \cdots + A_nu_{x+n-1}$$

Then u_xz^n is the general term in the expansion of

$$f(x)/(1 - A_1z^n - A_2z^{n-1} - A_3z^{n-2} - \cdots - A_nz)$$

and if α,β,γ,... roots of

$$y^n = A_1 + A_2y + A_3y^2 + \cdots + A_ny^{n-1}$$

then 1/α,1/β,1/γ,... roots of the above denom, with factors, besides some constant C, of (z - 1/α), and so on. Assume roots unequal. Then the fraction takes form:

$$f(z)/B(1 - \alpha z)(1 - \beta z)\cdots$$

with partial fractions

$$C_1/(1 - \alpha z) + C_2/(1 - \beta z) + \cdots$$

The coeff of z^n is found by expanding these. Therefore the general term of the recurring series is

$$u_x = C_1\alpha^x + C_2\beta^x + \cdots$$

$$\therefore \qquad \frac{u^{x+1}}{u^x} = \frac{C_1\alpha + C_2(\beta/\alpha)^{x+1} + C_3(\gamma/\alpha)^{x+1} + \cdots}{C_1 + C_2(\beta/\alpha)^x + C_3(\gamma/\alpha)^x + \cdots}$$

Let the max root = α, then x→∞, α = L u_{x+1}/u_x when x = ∞. To converge to the greatest root of

$$y^n = A_1 + A_2y + A_3y^2 + \cdots + A_ny^{n-1}$$

assume n arbitrary numbers for the first n terms with constants of relation being the above A_i. Then our u_{x+1}/u_x converges to α.

Example

$y^3 = -4 + 3y^2$
Assume terms 0,1,3 for A_i of -4,0,3 then series is

0 1 3 9 23 57 135 313 711 1593 3527 ...

The roots of this eqn are 2,2,-1 and this series converges to 2.
As in 9/3 = 3, ..., 3527/1593 = 2.214

We have here roots of α,β,γ where α = β. So our partial fraction takes form

$$C_1/(1 - \alpha z)^2 + C_2(1 - \alpha z) + C_3/(1 - \beta z) + C_4/(1 - \gamma z) + \cdots$$

u_x = coeff of $z^x = (C_1(x+1) + C_2)\alpha^x + C_3\beta^x + \cdots$

$$\therefore \quad \frac{u_{x+1}}{u_x} = \alpha \cdot \frac{(C_1(x+2) + C_2) + C_3(\beta/\alpha)^{x+1} + \cdots}{(C_1(x+1) + C_2) + C_3(\beta/\alpha)^x + \cdots}$$

and x→∞, this goes to α = 1/1

If we have i-roots, let $\beta = a + bi = R(\cos\theta + i \sin\theta)$
$R\cos\theta = a \quad R\sin\theta = b \quad \therefore\ R^2 = a^2 + b^2$
By De Moivre from DME, $\beta^n = R^n(\cos n\theta + i \sin n\theta)$
∴ α must be greater than R or α^2 greater than the product of the conjugate i-roots for this method to converge to the greatest root. This method can also be used to find the square roots of integers, using a method of Murphy's which I leave to your curiosity.

Fourier pursued this idea to the discovery of all roots of such a fn. But his method was somewhat flawed. Murphy shows how it may be, in some cases, corrected. Let me give you an example and you may pursue the theory if this tweaks your interest.

Example

$x^2 = 6x - 10$
Assume terms 1,2 for constant relations -10,6 then series is

1 2 2 -8 -68 -328

Take the last four terms. From product of extremes, -656. Subtract product of the means, 544, remainder is -1200. Using the penultimate 3 terms, from the product of the extremes, -136, take the square of the mean, 64, remainder is -200. Divide the former remainder by this: 6. This is the sum of the roots exactly and its half, 3, is the real part of the i-roots. The product of the last two terms is 2624. From this, subtract the square of -62 or 4624. Divide this remainder of -2000 by the former corresponding remainder, -200. The quotient, 10, is the product of the i-roots. Therefore the roots are 3 ± i. If that doesn't tweak your interest, you need to take a long hard look at yourself.

Newton's Method of Approximation

Let α be an approximate value of a root of φx = 0. Then α+h is the correct value with h very small.

$$\therefore \quad \varphi(\alpha) + \varphi'(\alpha)h + \varphi''(\alpha)h^2/2! + \cdots = 0$$

As h is very small, we have, very nearly (and I find that phrase endearing)

$\varphi(\alpha) + h\varphi'(\alpha) = 0$ OR $h = -\varphi(\alpha)/\varphi'(\alpha)$

But let this h be h_1, wash, rinse, repeat, and the h_i converge very rapidly. So we have

$\alpha_1 = \alpha - h_1$ $\alpha_2 = \alpha_1 - h_2$ and so on

If we denote $F(\alpha) = \alpha - \varphi(\alpha)/\varphi'(\alpha)$ we have

$\alpha_1 = F(\alpha)$ $\alpha_2 = F(\alpha_1) = F\bullet F(\alpha)$ $\alpha_3 = F\bullet F\bullet F(\alpha)$

and this series of compound fns converges on the root. Conversely, if this converges to α then

$F^n(\alpha) = \alpha$ $F^{n+1}(\alpha) = \alpha$ ∴ $F(\alpha) = \alpha$

But $F(\alpha) = \alpha - \varphi(\alpha)/\varphi'(\alpha)$ ∴ $\varphi(\alpha)/\varphi'(\alpha) = 0$. So if the roots are x_i [1-n] then

$\varphi'(\alpha)/\varphi(\alpha) = 1/\alpha\text{-}x_1 + 1/\alpha\text{-}x_2 + \cdots$

∴ α is a root

By Sturm's Theorem, one could find two limits, a,b, to a root. Then in Newton's Method, h could begin as (a - b)/n where n divides [a,b] into a large number of parts. This was developed into a method by which Fourier, by repeated application, could determine a root of $x^3 - 2x - 5 = 0$ to be:

2.09455148154232659148238654057930

although what he gained here, beyond bragging rights, is more than I could tell you. Simpson, who also had a great tolerance for tedium, further improved this method. But a method of numerical approximation, human or digital, does not reveal the laws governing those approximated roots. So again, I leave these methods to your curiosity, Simpson is able to approximate, in his way, the products of i-roots, which gives you something to be curious about. I should also mention that Fourier's method uses φx, φ'x, φ"x together with Euclid's Algorithm, but not in any new way that we haven't seen.

Method of Continued Fractions

If φx = 0 is an algebraical eqn with real roots, we can use the following method to find its positive roots and find the negative roots by making x = -y and then positive y's give negative x roots. Divide φx by φ'x ≡ V_1. Change the sign of the remainder ≡ V_2. Divide φ'x by V_2 and change sign of remainder for V_3. Divide V_2 by V_3, wash, rinse, repeat, until you get to a constant remainder ≡ V_m. If we make x = 0 we reduce φx and all V_i to last terms and note their sign. Then x = +∞ and note the signs of the first terms. Number of alternations of sign is number of positive real roots. Sub 1, 10, 100, 1000, … for x until you have as few changes of sign as for the first terms. The number of alternations lost in this are roots on [1,10], [10,100], and so on. Say there is a root on [10,100]. Repeat subbing x with 20, 30, ..., 90. Say the root(s) are on [30,40]. Repeat with 31, 32, ..., 39. In the end we have the integer parts of all the positive roots. Let p be an integer here. The changes of sign on φ and V_i from p to p+1 give you how many roots are here. Make x = p + 1/y and as φx is n° we have

$$\varphi(p)y^n + \varphi'(p)y^{n-1} + \varphi''(p)/2! \cdot y^{n-2} + \cdots + \varphi^{(n)}(p)/n! = 0$$

We want to approximate the values of y on (p,p+1). So y must be greater than 1. We can find the integral parts of y, just as we did for x. Then we will have, for each y we seek to approximate, a q and q+1 where y ∈ [q,q+1]. To separate very close roots, wash, rinse,repeat with y = q + 1/z, z = r + 1/u, and so on.

We then, for each root, have a convergent continued fraction, p: q, r, s, ... Lagrange extends this method to a laborious one for i-roots. But, hey, this is all laborious. And Murphy notes that, for approximation, the method of recurring series, with properly chosen arbitrary terms, is the least laborious of any method for approximating roots.

For p, q, r, s, ..., we know, from Chrystal, that we can calculate these. Let the convergents be

1/3 3/10 10/33 33/109 109/360 360/1189 ⋯

If we represent the convergent value of this series as x and use it in the first convergents, we have

$$x = 1/(3+x) \quad \text{OR} \quad x^2 + 3x = 1$$
$$\therefore \quad x = -3/2 \pm \sqrt{(13/4)}$$

But we can only use positive values ∴ x = ½(√13 - 3). Murphy works out three examples of this long method if you are curious. At this point, Murphy considers continued fractions, going over some of what Chrystal gave us with an earlier, more awkward notation. He does show how to convert an algebraic expression into a CF. Let me give you his first example. Following his thinking here is a rewarding entertainment, in my opinion.

Example

Convert $((x + 1)/x)^2$ into a CF: x > 2
Let x = 2y ⇒ $4y^2$ contained once in $4y^2 + 4y + 1$ as 2 > largest root of $4y^2 + 4y - 1 = 0$
∴ x ≥ 2 ∴ $4y^2 > 4y + 1$ ∴ $8y^2 > 4y^2 + 4y + 1$
∴ 1 is the first quotient, remainder 4y + 1
4y + 1 is not contained y times in $4y^2$
Trying y-1 as quotient ⇒ positive remainder 3y + 1
3y + 1 contained once in 4y + 1, remainder y
y contained 3 times in 3y + 1, remainder 1 and this terminates the operation OR

[cont'd]

$4y^2$) $4y^2 + 4y + 1$ (1
$\underline{4y^2}$
$4y + 1$) $4y^2$ ($y - 1$
$\underline{4y^2 - 3y - 1}$
$3y + 1$) $4y + 1$ (1
$\underline{3y + 1}$
y) $3y + 1$ (3
$\underline{3y}$
1) y (y
0

$\therefore x = 2y \Rightarrow ((2y + 1)/ 2y)^2 = 1$: y-1, 1, 3, y
But when x is odd this becomes $(2y/(2y-1))^2$ and by the same method:

$4y^2 - 4y + 1$) $4y^2$ (1
$\underline{4y^2 - 4y + 1}$
$4y - 1$) $4y^2 - 4y + 1$ ($y - 1$
$\underline{4y^2 - 5y + 1}$
y) $4y - 1$ (3
$\underline{3y}$
$y - 1$) y (1
$\underline{y - 1}$
1) $y - 1$ ($y - 1$
0

$\therefore (2y/(2y - 1))^2 = 1$: y - 1, 3, 1, y - 1

Murphy continues this by going over most of what Chrystal covered with CF. If you are interested, you might enjoy seeing Murphy's viewpoint of these ideas as it differs from Chrystal's. Let's follow him further into using CF to solve eqns.

Thm. 7.5. In soln by CF, the transformed eqns after the first few will have opposite signs for first and last terms.

Proof

$\varphi(x) = 0$ with λ max int < root of φx
$x = \lambda + 1/y \Rightarrow$ eqn in y has as many pos. roots > 1 as there are values of x ·|·$(\lambda,\lambda+1)$
It is possible that 2+ values of y are between some λ' and $\lambda'+1$
Then we make $y = \lambda' + 1/z$ with $\varphi(z)$ having its numbered roots symmetrically > 1
As we wash, rinse, repeat, we reach an eqn with only one root between any two successive integers:

$$au^m + bu^{m-1} + cu^{m-2} + \cdots + k = 1 \qquad [1]$$

Then, with $u = s + 1/w$, s being the nearest integer below one of [1]'s roots, F(w) can have only one pos. root > 1 and then all the consecutive transformed eqns will be in the same condition.

[proof cont'd]

Take F(w) = 0, l max int. below the only positive root > 1, and w = l + 1/t. Then

$$F(l+1/t) = 1/t^m(F(l)t^m + F'(l)t^{m-1} + F''(l)/2!\cdot t^{m-2} + \cdots + F^{(m)}(l)/m!) = 0$$

∴ 1st term $F(l)t^m$, last term $F^{(m)}(l)/m!$ must have opposite signs because
by hyp. ∃!root of F(w) = 0 ∈ (l,+∞)
∴ F(l), F(+∞) have opposite signs OR F(l + 1/t) has contrary signs for F(0), F(+∞)
∴ $t^m F(l + 1/t) = F^{(m)}(l)/m!$ when t = 0 and this has a sign contrary to F(l)
This same reasoning holds for all succeeding transforms. ■

I like Murphy's conversion of a CF to a series. Let p_i/q_i [1-n] be the convergents and P/Q the value of the infinite series.

$p_2/q_2 - p_1/q_1 = (p_2q_1 - p_1q_2)/q_1q_2 = -1/q_1q_2$
$p_3/q_3 - p_2/q_2 = 1/q_2q_3$
$p_4/q_4 - p_3/q_3 = -1/q_3q_4$
…
$p_n/q_n - p_{n-1}/q_{n-1} = (-1)^{n-1}/q_{n-1}q_n$
∴ $p_n/q_n = p_1/q_1 - 1/q_1q_2 + 1/q_2q_3 - \cdots = (-1)^{n-1}/q_{n-1}q_n$
∴ $P/Q = p_1/q_1 - 1/q_1q_2 + 1/q_2q_3 - \cdots$

Because the terms monotonically diminish and alternate in sign the error of taking n terms is less than the (n+1)th term. He adds "*in some cases, it is convenient to take some of the partial denoms as negative.*" Isn't that interesting?

On Peculiar and Infinite Equations

We are on the last seven pages of Murphy's text. This last bit is interesting to me as it draws together so many disparate ideas. This is the first time I have come across such a thing and I suspect it reveals something of Murphy's individual interests. Let's consider some "peculiar eqns" by forming an eqn whose roots are ∀z ∈ **Z** OR

$F_n(x) = Cx(x - 1)(x - 2)\cdots(x - n)(x + 1)(x + 2)\cdots(x + n)$
∴ $F_n(x + 1) = C(x + 1)x(x - 1)\cdots(x - n + 1)(x + 2)(x + 3)\cdots(x + n + 1)$
∴ $F_n(x + 1) = (x + n + 1)/(x - n) \cdot F_n(x)$

Let φx be the eqn sought as n → ∞. Then (x + n + 1)/(x - n) → -1

∴ $\varphi(x + 1) = -\varphi x$

To solve this (recall from DME) let $\varphi x = C\varepsilon^{mx}$

∴ $C\varepsilon^m \cdot \varepsilon^{mx} = -Ce^{mx}$
∴ $\varepsilon^m = -1$
∴ $m = \pm i\pi, \pm 3i\pi, \pm 5i\pi, \ldots$

This particular form, the sum which gives the general for of φx which coincides with the above eqn of differences are these:

Asinπx + Bcosπx
A'sin3πx + B'cos3πx
A"sin5πx + B"cos5πx
…

The choice to use an eqn of differences ensures that the roots of these fns are in A.P. But we need our 0 value as well. And the sines of odd multiples of πx vanish for fractional x, not just integer x. And we are here left with Asin(πx) = φx.

if we wanted φx to have roots 1/2, 3/2, 5/2, …, -1/2, -3/2, -5/2, …
then our φx = Acos(πx).

Now we want some φx with roots 1, 1/2, $1/2^2$, $1/2^3$, …, $1/2^n$ or

$$\varphi x = (x - 1)(x - 1/2)(1 - 1/2^2)\cdots((x - 1/2^n)$$

Sub 2x for x, separate numerical factors of x and

$$\varphi(2x)/2^{n+1} = (x - 1/2)(1 - 1/2^2)\cdots(x - 1/2^n)(x - 1/2^{n+1})$$
$$\therefore \quad (x - 1)\varphi(2x)/2^{n+1} = (x - 1/2^{n+1})\varphi x \qquad [1]$$

So φx has form

$$x^{n+1} - a_1x^n + a_2x^{n-1} - a_3x^{n-2} + \cdots$$
$$\therefore \quad (x-1/2^{n+1})\varphi x = x^{n+2} - (a_1/2 + 1)x^{n+1} + (a_2/2^2 + a_1/2)x^n - (a_3/2^3 + a_2/2^2)x^{n-1} + \cdots$$
$$\therefore \quad \varphi(2x)/2^{n+1} = x^{n+1} - a_1/2\cdot x^n + a_2/2^2\cdot x^{n-1} - a_3/2^3\cdot x^{n-1}$$
$$\therefore \quad \text{LHS } [1] = x^{n+2} - (a_1/2 + 1)x^{n+1} + (a_2/2^2 + a_1/2)x^n - (a_3/2^3 + a_2/2^2)x^{n-1} + \cdots$$

Comparing the general coeff of both sides of [1]:

$$a_p + (a_p - 1)/2^{n+1} = a_p/2^p + a_{p-1}/2^{p-1}$$
$$\therefore \quad a_p(2^p - 1) = a_{p-1}(2 - 1/2^{n-p+1})$$
$$\therefore \quad a_1 = 2 - 1/2^n \quad a_2 = (2 - 1/2^{n-1})/(2^2 - 1)\cdot a_1$$
$$a_3 = (2 - 1/2^{n-2})/(2^3 - 1)\cdot a_2$$ and so on and we have our φx.

Now let's form an eqn with roots $-\alpha_1$, $-(\alpha_1 + \alpha_2)$, $-(\alpha_1 + \alpha_2 + \alpha_3)$, …, $-\sum\alpha_n$ [1-n]
We form the product

$$(x + \alpha_1)(x + \alpha 1 + \alpha_2)\cdots(x + \alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

and this takes the form

$$x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_mx^{n-m} + \cdots + A_{n-1}x + A_n$$

The general coeff here is A_m being the sum of the products with m factors.

$$A_1 = n\alpha_1 + (n-1)\alpha_2 + (n-2)\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$$

A_2 consists of products $\alpha_i\alpha_j$ and pure powers like α_i^2, α_j^2. The general form of the first class of terms is $\alpha_p\alpha_q$ and we need its coeff, which we denote $(\alpha_p\alpha_q)$. Let p<q and no factor before $(x + \alpha_1 + \alpha_2 + \cdots + \alpha_p)$ need be considered. So we care about these factors:

$$\begin{aligned}&x + \alpha_1 + \cdots + \alpha_p\\ &x + \alpha_1 + \alpha_p + \alpha_{p+1}\\ &\cdots\\ &x + \alpha_1 + \cdots + \alpha_p + \cdots + \alpha_q{}^*\\ &x + \alpha_1 + \cdots + \alpha_p + \cdots + \alpha_q + \alpha_{q+1}\\ &\cdots\end{aligned}$$

Now if α_{p+1} were placed at the *, the combs of α_p with α_q and α_{q+1} would be the same.
∴ $(\alpha_p\alpha_q) - (\alpha_p\alpha_{q+1})$ is the number of combs of one term at the * with the terms in the column of α_p **except** that α_p with the *.
∴ The number of terms in that column minus one will be n-p.
So let Δ denote the finite difference as q increases by unity
∴ $(\alpha_p\alpha_q) = -(n - p)$
∴ $(\alpha_p\alpha_q) = (n - p)(C - q)$ where C is independent of q
If q = n then, observing the column of α_p:

$$(\alpha_p\alpha_q) = (n - p)(n - q + 1)$$

Denoting the coeffs of powers, say α_p^2 as (α_p^2), they are not affected by zeroes, as in $\alpha_1 = 0$ or $\alpha_n = 0$.
∴ $(\alpha_p^2) = (1 + \alpha_p)^{n-p+1}$ which algebraically means $(n - p)(n - p + 1)/2!$
∴ $A_2 = n(n-1)/2!\cdot\alpha_1^2 + (n-1)(n-2)/2!\cdot\alpha_2^2 + (n-2)(n-3)/2!\cdot\alpha_3^2 + \cdots + (n-1)(n-1)\alpha_1\alpha_2 + (n-1)(n-2)\alpha_1\alpha_3 + (n-1)(n-3)\alpha_1\alpha_4 + \cdots + (n-2)(n-2)\alpha_2\alpha_3 + (n-2)(n-3)\alpha_2\alpha_4 + \cdots + (n-3)(n-3)\alpha_3\alpha_4 + \cdots$
and this can be Sym. pursued for all A_i if your curiosity maliciously compels you to pursue it. Now let's consider what Murphy calls "infinite eqns".

Given $1 + x^2 + x^3 + \cdots = 0$ its derivative can be considered as

$$1 + nz + n(n-1)z^2 + n(n-1)(n-2)z^3 + \cdots = 0$$

which is a finite eqn for $n\in\mathbf{N}$ and is infinite if $z = x/n$ as $n\to\infty$. Let $z = 1/y$ and the derivative becomes

$$y^n + ny^{n-1} + n(n-1)y^{n-2} + n(n-1)(n-2)y^{n-3} + \cdots = 0$$

Let LHS of this be u and its derivative u' then

$$u = y^n + u' = 0$$

Let the real roots, if any, of u = 0 be α,β,γ,... in descending magnitude. If y = α in u' then u' > 0 and $y^n<0$ which if n is even is impossible and no real roots. But if n is odd, we have real roots and α makes $y^n<0$.

∴ α and any other real roots are negative

But β makes u'<0 ∴ $y^n>0$ ∴ y has one real root

If y = 0 in u, u equals its last term which is positive.

If y = -√n, u is negative ∴ y ∈ (0,-√n) ∴ z ∈ $(-n^{-1/2}, -\infty)$

This series is 1/1-x which only vanishes for x = +∞.

Consider $x - x^3/3! + x^5/5! - \cdots = 0$ and its derivative

$$nz - n(n-1)(n-2)/3!\cdot z^3 + n(n-1)(n-2)(n-3)(n-4)/5!\cdot z^5 - \cdots = 0$$

which terminates for n∈**N**. Let x = nz and n→∞. This eqn is

$$(1 + iz)^n - (1 - iz)^n \quad \text{OR} \quad ((1 + iz)/(1 - iz))^n = 1$$

Let z = tanθ ∴ $((\cos\theta + i\sin\theta)/(\cos\theta - i\sin\theta))^z = 1$ OR cos2nθ + isin2nθ = 1

∴ 2nθ = (2n - 2)π for n∈**N**

For z = 0 ±tan(π/n), ±tan(2π/n), ... n→∞ ⇒ x = 0 at zπ for z∈**Z**

Consider $1 - x + x^2/2! - x^3/3! + \cdots = 0$

Derivative $1 + nz + n(n-1)/2!\cdot z^2 - n(n-1)(n-2)/3!\cdot z^3 + \cdots = 0$ OR $(1 - z)^n = 0$

where root is z = 1 ∴ x = ∞ and !∃ finite root.

Murphy's last example is

$$1 - x/1^2 + x^2/(1^2\cdot 2^2) - x^3/(1^2\cdot 2^2\cdot 3^2) + \cdots = 0$$

with derivative

$$1 - n/1 \cdot (n-1)/1\cdot z + n(n-1)/2! \cdot (n+1)(n+2)/2! \cdot z^2 - \cdots = 0$$

where all roots are real and positive. And all are on (0,1). The difference of two separate roots is of the order 1/n. So the limits of all roots x_i are found by x = n ∀n∈**N** and by observing alternation of sign. When x < 1 all terms in parens are positive or

$$(1 - x) + x^2/2^2 (1 - x/3^2) + x^4/(2^2 3^2 4^2) (1 - x/5^2) + \cdots$$

so there is no root ≤ 1. When x = 2, the first three terms vanish and the series becomes

$$-2/3^2(1 - 2/4^2) - 2^3/(3^2 4^2 5^2) (1 - 2/6^2) - \cdots < 0$$

So there is a real root on (1,2). When x = 3, the first five terms are negative and the remainder, taken in pairs are negative. So there is no root on [2,3]. And the same for x = 4,5,6,7,8.

If we call this fn and its derivative u and u', then u' = u_1 with its u". And u" is u_2 and so on. Then

$$u_m - (-x)^m u^{(m)} = 0 \text{ where } u^{(m)} \text{ is the mth derivative}$$

Or if m = 1 then $u_1 + xu' = 0$. The substitution of roots of u=0 in u' would produce a series of alternating signs and as x > 0, this is true of all u_i. Murphy notes that the definite integrals of u multiplied by other fns relate to the form electricity takes in solid bodies. And there he ends.

8. Resolution of Algebraic Equations

Let's locate ourselves in time and space. Galois's papers were published by Liouville in 1846. Hargreave's *The Resolution of Algebraic Equations* was published in 1866. It was published by the Cambridge mathematician George Salmon. Salmon had read the manuscript and disagreed with Hargreave about its substance. Hargreave, already in ill health exacerbated by the effort of this book on top of his work as a judge, rewrote it and died just after he gave Salmon the last of the appendices. Salmon, who now agreed with the text, saw that it was published. Todhunter recommends the text in his book on Equation Theory, although, to my mind, he is a tad sly about it. If you look into what is known about Hargreave, most bios cite that he died while this book was being written and are unaware of its existence.

On a personal note, I had read only Salmon's preface to Hargreave up until the point of having written the first 190 pages of this manuscript. So any resemblance of my thought to Hargreave's up to there is purely coincidental. And one more historical thing: Todhunter thought that one could spend too much effort on Analytical Geometry, which was Salmon's speciality. Salmon wrote several volumes on that topic, each of which would keep a door open in a high wind. This might explain Todunter's slyness. Now here comes Hargreave:

Introduction

"To resolve an equation" differs in a logical sense from all other problems in mathematics. If we conform to the laws of the ground-truths -- of number and operations -- the resolution proves itself. If x_1 and x_2 are roots of a quadratic, then

$$\tfrac{1}{2}(x_1 + x_2 + \sqrt{(x_1^2 + x_2^2 - 2x_1x_2)}) \qquad [1]$$

identically represents the resolution of any quadratic to its roots. The existence of [1] proves its truth. It matters not at all how we arrive at [1]. Our reasoning will be of an *a priori* nature. Whatever steps lead us to [1] are legitimated by [1].

The same is true of eqns of degrees 3 and 4. Different human beings arrived at different solutions for their three and four roots. There is obviously a practical limit to this approach. All such methods so far have been our "solution by radicals" and all have been shown to be equivalent. But there is a *necessity of not assuming that all modes of solution necessarily conduct to the same result, so far as relates to the* ***form*** *of resolution*.

It has been shown that for eqns of degree 5 and higher the existing **form** of soln by radicals cannot possibly succeed.

The only method left to us is to proceed by steps; that is, in effect, so soon as we have resolved the problem in any one case, say the lowest, to endeavour to make use of this step, in order, if possible, to mount to the next step; and to do this, as far as possible, upon some general principle, the applicability or non-applicability of which is shown by the success or failure of the attempt to apply it.

If it is true of some number m that an eqn of degree m-1 is algebraically resoluble while one of degree m is not, it must be that the second eqn is not of the same species as the first. So there must be some way of stating the first algebraically which cannot be used to state the second.

Yet nothing of this kind has manifested itself in all the infinite researches which have been made into this problem. The cases which have been resolved are known by that fact to be resoluble, but not very clearly, in any other way. The cases which are not resoluble are demonstrated to be irresoluable not by anything really peculiar to these cases of such a nature as to distinguish them from the others, but by an elaborate inquiry into all the possible modes of algebraic expression, and an exhaustive proof that no one of them can qualify itself to be the expression of the root.

Notation, Definitions, and Elementary Theorems

The general algebraic eqn is a **quantic** φ_n:

$$x^n - na_1x^{n-1} + \tfrac{1}{2}n(n-1)a_2x^{n-2} - \cdots \pm na_{n-1}x \mp a_n = 0$$

(1p) is the resultant of φ_1 and φ_p from the subbing of a_1 for x in φ_p and changing the sign. If $t = x - a_1$, the quantic takes form:

$$t^n - (11)t^{n-1} - \tfrac{1}{2}n(n-1)(12)t^{n-2} - \cdots - n(1(n-1)t - (1n) = 0 \qquad [2]$$

$(11) = 0$
$(12) = a_1^2 - a_2$
$(13) = 2a_1^3 - 3a_1a_2 + a_3$
$(14) = 3a_1^4 - 6a_1^2a_2 + 4a_1a_3 - a_4$
$(15) = 4a_1^5 - 10a_1^3a_2 + 10a_1^2a_3 - 5a_1a_4 + a_5$
…

Example

$x^2 - 2x + 1 = 0 \qquad a_1 = 1 \;\; a_2 = 1$
$(12) = (x+1)^2 - 2(x+1) + 1 = x^2 + 2x + 1 - 2x - 2 + 1 = x^2$
$(11) = 0 \quad (12) = a_1^2 - a_2 = 1^2 - 1 = 0$

So these (xy) are **numbers** and Hargreave uses them extensively. The (1p) comes from $t = x - a_1$ ∴ (2p) comes from $t = x - a_2$. In the quantic, x is the **argument** (arg), a_i [1-n] are the **coefficients** (coeff), x_i [1-n] are the roots. Transformed into [2], the arg is t or $x-a_1$ and the coeffs (12) to (1n) are n-1 in number. This is a **linear transformation** (lin.trans.) and (12) … (1n) are sym. fns of (x_1-a_1), (x_2-a_1), …, (x_n-a_1).

We could use other resultants, as in (23), the resultant of φ_2 and φ_3.

S_p denotes a sym. fn of roots x_i [1-n] of pth degree (n=p) ∴ also a qfn of a_i [1-n].

R_p is a qfn of p° which may or may not be sym.

The **discriminant** (dscr) of a quantic as well as every combination of roots or coeffs whose vanishing shows a condition of a system or systems of equations among the roots can be expressed in terms of the n-1 quantities (12), (13), …, (1n).

Discriminants

Quadratic	(12)
Cubic	$4(12)^3 - (13)^2$
Quartic	$81(12)^4(14) - 54(12)^3(13)^2 + 18(12)^2(14)^2 - 54(12)(13)^2(14)$ $+ 27(13)^4 + (14)^3$
Quintic	$3456(12)^5(15)^2 + 11520(12)^4(13)(14)(15) - 6400(12)^4(14)^3$ $- 5120(12)^3(13)^3(15) + 3200(12)^3(13)^2(14)^2$ $- 1440(12)^3(14)(15)^2 + 2640(12)^2(13)^2(15)^2$ $+ 448(12)^2(13)(14)^2(15) - 2560(12)^2(14)^4$ $- 10080(12)(13)^3(14)(15) + 5760(12)(13)^2(14)^3$ $+ 120(12)(13)(15)^3 - 160(12)(14)^2(15)^2 + 3456(13)^5(15)$ $- 2160(13)^4(14)^2 + 360(13)^2(14)(15)^2 + 640(13)(14)^3(15)$ $- 256(14)^5 + (15)^4$

Note that the quartic dscr can take form

$$27((12)^3 + (12)(14) - (13)^2)^2 - (3(12)^2 - (14))^3$$

which resembles the cubic. If the quantic can be algebraically resolved by lin.trans. the rational part of the root is a_1 and the residue will be the n-1 fns (12), (13), …, (1n). This is only necessarily true if resolvable by our lin.trans. The x_i are independent roots or symbols, the a_i independent coeffs or parameters. No relation between them or character of them is assumed.

A **transformation** (trans.) applied to φ_n gives us one or more n° eqns with relations between coeffs and/or roots. These will be eqns in z with coeffs b_i. Each eqn is denoted λ_n and the resultant of λ_1 and λ_p will be denoted (p1) to distinguish then from (1p) for eqns in x. If we consider with φ_n a subsidiary eqn of lesser degree or **resolvent** (rsvt), it will be a fn of y with resultants denoted (I·II), (I·III), and so on.

∑ will be used as in our sigma notation from Chrystal of perms and the following sym. fns of roots will be abbreviated by underlining and these can be expressed in terms of x_i or a_i:

$\underline{pq} \equiv (n-1)\sum(x^p)\sum(x^q) - n\sum(x^px^q)$ $p \neq q$

$\underline{pp} \equiv (n-1)(\sum(x^p))^2 - 2n\sum(x^px^p)$

$\underline{pqr} \equiv (n-1)(n-2)\sum(x^p)\sum(x^q)\sum(x^r) - n^2\sum(x^px^qx^r)$ p,q,r different

$\underline{ppq} \equiv (n-1)(n-2)(\sum(x^p))^2\sum(x^q) - 2n^2\sum(x^px^px^q)$ $p \neq q$

$\underline{ppp} \equiv (n-1)(n-2)(\sum(x^p))^3 - 6n^2\sum(x^px^px^p)$

Consideration of Operations

We consider addition, multiplication, prime powers, and roots of same. Addition tells us what is positive and what is negative. Multiplication tells us what is whole and what is fractional. And powers tell us what is rational or irrational.

We will make the following important distinction. If p prime and P an expression which cannot be obtained by raising any expression to the pth power then $\sqrt[p]{P}$ is an irrational or surd expression. We know that it has p values but they are indistinguishable as all require the symbol $\sqrt[p]{}$. But if the p values of $\sqrt[p]{P}$ are algebraically distinguishable then $\sqrt[p]{P}$ is not irrational but of the form $\sqrt[p]{Q^p}$ where we can write out the separate values of ω^iQ where ω^i is a root of unity. Here P is the product of ωQ, ω^2Q, ω^3Q, …, ω^pQ. In the first case, no Q exists and using $\sqrt[p]{P}$ in place of Q merely maintains ambiguity.

It may well be supposed that this is merely a metaphysical distinction. Whether this be so or not, the distinction is a real one; and it constitutes probably the clearest mode of distinguishing between two notions, a rational expression and an expression which is identically and necessarily surd.

Example

Given a cubic, we find a product of two qfn of the roots, each 3°, which is a perfect cube of a sym. fn 2° of the same roots. We also know that the same proposition in the same terms is applicable to two other cubics in which the 2° sym. fn is in the one case ω times and in the other ω^2 times what is in the original. The **important** conclusion is that **each** of the qfns whose product is a perfect cube is itself also a perfect cube. Also, we could not otherwise divide each factor into three **distinguishable** subordinate factors. More explicitly, let these two fns of the roots have the product $(12)^3$ which is $(x - a_1)^3$ and this product is unchanged by using $\omega(12)$ and $\omega^2(12)$. Then each fn is a perfect cube and therefore a perfect cube of a linear fn of the roots. This is the only way we can have this ω, ω^2 substitutability. The only other way to compose this product is with factors R_1^2, R'_1 and R_1, R'^2_1 with subfactors of $\sqrt[3]{(R_1^2R'_1)}$ which remain indistinguishable. This principle of distinction will be further developed.

Systems of Equalities, Critical Functions

Conditions among roots can be expressed by roots or coeffs. If we wish to impose the linear condition

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = 0$$

we multiply together the n! eqns which include this and all its changes by transpositions of the roots. The result is a sym. fn of the roots expressed as a qfn in terms if its coeffs. If the A_i are not all different, the degree of the condition is a submultiple of n!.

Most importantly we consider the condition of two equal roots, expressed by equating to zero the product of all $x_p - x_q$ which are n(n-1) in number. And this condition is a perfect square. Expressed in coeffs, it is the resultant of φ_{n-1} and φ_n, which is in turn the dscr of φ_n. The condition of three equal roots are

1) 2 equal roots
2) φ_n' has two equal roots $\equiv$ φ and φ'' have a common root

In a cubic, the condition of two equal roots is

$$(23) = 0 \wedge (12) = 0 \quad \text{OR} \quad (23) = 0 \wedge (13) = 0$$

or, as $(23) = (13)^2 - 4(12)^3$

$$(12) = 0 \wedge (13) = 0$$

In a cubic, three equal roots can be expressed by linear relations among the roots. As (23) is a perfect 6° square, the condition of 2 equal roots is expressible as a 3° fn. And as (13) is 3°, we can, with suitable N, present both as

$$N(13) + \sqrt{(23)} = 0$$

which expressed by the roots is a two-valued qfn. The value of N must ensure

$$(12) = 0 \wedge ((23) = 0 \vee (13) = 0)$$

By adding these, they are equivalent to (13) = 0. If N = 1, they are equivalent to $(12)^3 = 0$ which is (12) = 0 by multiplication. We have two qfns of x_i [1-3] whose product is a perfect cube of (12). So, as in our example above, each qfn is a perfect cube of a linear fn of the roots. We then have three linear fns of the roots, each cubed equaling $(13) + \sqrt{(23)}$ and three others, which cubed equal $(13) - \sqrt{(23)}$ and taking one from each set and adding them together is a sym. fn of the roots. The only linear fns which will work for this are

$$A(x_1 + \omega x_2 + \omega^2 x_3) \quad \text{and} \quad A(x_1 + \omega^2 x_2 + \omega x_3)$$

and also each of these multiplied by ω and ω^2.

So the condition of three equal roots in a cubic is

$$x_1 + \omega x_2 + \omega^2 x_3 = 0 \quad \wedge \quad x_1 + \omega^2 x_2 + \omega x_3 = 0$$

Now consider the quartic. Two equal roots needs (34) = 0 and three equal roots requires

$$(34) = 0 \;\wedge\; (23) = 0$$

(34) = 0 is a 12° fn of the roots and a perfect square. Its square root is homogeneous with (23) = 0 or any other 6° condition.

$$\begin{aligned}(34) &\equiv 27((12)^3 + (12)(14) - (13)^2)^2 - (3(12)^2 - (14))^3 \\ &\equiv 27((23) - (12)(3(12)^2 - (14)))^2 - (3(12)^2 - (14))^3\end{aligned}$$

So for two equal roots we need two of the following conditions met:

$$\begin{aligned}&3(12)^2 - (14) = 0 \\ &(12)^3 + (12)(14) - (13)^2 = 0 \\ &(34) = 0\end{aligned}$$

With suitable N, the two become

$$N(\,(12)^3 + (12)(14) - (13)^2\,) \pm \sqrt{(34)}$$

Their sum is one condition and their product, if $N = \sqrt{27}$, is the other condition. Sym. with the cubic, the linear conditions of three equal roots where the R's are 2° qfns of x_i [1-4] are

$$R_2 + \omega R'_2 + \omega^2 R''_2 \quad \wedge \quad R_2 + \omega^2 R'_2 + \omega R''_2$$

The Quadratic

The simple application of the lin.trans. **is** the resolution of the **quadratic** (quad) of

$$\begin{aligned}&(x - a_1)^2 - (12) = 0 \\ &\therefore\; x = a_1 \pm \sqrt{(12)} \qquad\qquad [1]\end{aligned}$$

which to determine x_1 and x_2 is equivalent to

$$\tfrac{1}{2}(x_1 + x_2 + \sqrt{(x_1^2 + x_2^2 - 2x_1x_2)})$$

The quad admits of no other transformation so [1] is the only form of the root. Its rational part is a_1. The irrational part is a fn of (12). The quad can undergo only one change of system with respect to the roots. So there can be only one radical in its roots and that one must be $^2\sqrt{}$. Its soln is therefore unique.

Recall our notation S_p for sym. fns of the roots. The roots of a quad can be expressed as $S_1 + \sqrt{S_2}$ and therefore S_2 is a perfect square. So we must always be able to express the two roots as distinguishable. If S_2 were not a perfect square, there would be no possibility of algebraically distinguishing the two forms of expression. So here, the $\sqrt{}$ must go away. *In this consideration, we see the first step towards the views embodied in Abel's Theorem* (regarding the expression of roots with respect to coeffs).

In our process, we have inverted the ordinary algebraic operation. If, in the quad, we sub $2a_1t$ for x, the quad becomes

$$t^2 - t = -a_2/4a_1^2 = \text{(say) } v$$
$$\therefore \ \varphi t = v$$

So we need to solve $t = \varphi^{-1}v$ where φ is a fn only of t with no constant term, and t is a multiple of x. OR

Given $v = t^2$, what fn of v is t?

As a side note, before reading this, I had thought one could say **everything** about quadratics in about ten pages of a text like DME. But Hargreave's viewpoint is so original and expansive, I think another couple of pages would now be required to do justice to quadratics.

The Cubic

Our view of the quad suggests that the rational part of the cubic's roots may be a_1 and the irrational parts fns of (12) and (13). We would like to find fns of the cubic roots which are the roots of a quad whose coeffs are rational expressions (qfns) of the cubic's coeffs. These quads should have specific relations to the cubic's roots and might aid in their resolution.

An algebraic expression of any quantic's roots should mirror changes caused in the system by equal roots. So here any changes in the cubic system should be mirrored in our desired quads. If a cubic has three equal roots, each is a_1 and the irrational part vanishes. This causes any radicals to disappear. So our quads, as expressions of irrational parts, would disappear. If the cubic has two equal roots, the irrational part of its roots would change in form, where a quad would go from two roots to one. We then need a quad whose coeffs are rational fns of the cubic's coeffs (12) and (13) and whose dscr (discriminant, remember?) is the same as the cubic's.

A cubic's dscr is 6° in its roots and a quad's is 2°. If they are to be equal, the argument of any quad in y must be 3° and its coeffs of degrees 0,3,6. The most general form of this would be

$$y^2 - 2N_1(13)y + N_2(12)^3 + N_3(13)^2 = 0$$

We can let $N_3 = 0$ without loss of generality as it would vanish under a transform such as $y = t + N_4(13)$ and such transform affects neither the rationality of coeffs nor the dscr.

So we have

$$y^2 - 2N_1(13)y + N_2(12)^3 = 0$$

We need, to a numerical factor, equal dscrs for cubic and quad. The quad's is

$$N_1^2(13)^2 - N_2(12)^3$$

and the cubic's, from our table some pages back, is

$$4(12)^3 - (13)^2$$
$$\therefore\ N_2 = 4N_1^2$$

As N_1's value is immaterial, let it be unity. So our quad becomes

$$y^2 - 2(13)y + 4(12)^3 = 0$$

Its soln gives two values of y in terms of (12),(13) and therefore of a_i [1-3] which are known. By subbing x_i appropriately for (12),(13) or for a_i the y's are sym. fns of x_i:

$$y_1 = S_3 + \surd S_6$$
$$y_2 = S_3 - \surd S_6$$

which are rational, as S_6 is the dscr of the cubic and therefore a perfect square in terms of the roots. We see this rationality by comparing y expressed wrt y_1,y_2 and wrt x_1,x_2,x_3. The former is

$$½(y_1 + y_2 + \surd(y_1^2 - 2y_1y_1 + y_2^2))$$

and the latter, in our Sigma notation, is

$$2/27(\sum x)^3 - 1/3\sum x\sum(xx) + \prod x + \surd(N(x_1\text{-}x_2)(x_1\text{-}x_3)(x_2\text{-}x_3))^2)$$

where N is a known numerical factor.
∴ y_1+y_2 and y_1-y_2 can be expressed rationally in terms of x_i
∴ y_1 and y_2 are so expressible
∴ each y is a rational fn of x_1-a_1, x_2-a_1, x_3-a_1
The product of the 2 y's is a perfect cube of a qfn of the coeffs
∴ each y is a perfect cube
∴ each y is the cube of a linear fn of x_i - a_1 [1-3]

$$y_1^{1/3} = A_1(x_1 - a_1) + A_2(x_2 - a_1) + A_3(x_3 - a_1)$$
$$y_2^{1/3} = B_1(x_1 - a_1) + B_2(x_2 - a_1) + B_3(x_3 - a_1)$$

where the A_i,B_i come from our calculation by way of an extraction of the roots, which we have proven possible. These, combined with

$$(x_1 - a_1) + (x_2 - a_1) + (x_3 - a_1) = 0$$

determine the roots. We must be able to calculate y as a fn of a_i and of x_i. The a_i give us $y_1^{1/3}$ and $y_2^{1/3}$. The x_i give us the A_i,B_i. But here these could be inferred.

If we consider our success in finding the linear fns in $(x_i - a_1)$, we still have not a definite soln of the cubic. For our $y_i^{1/3}$, we have no guide to know which root of unity is our numerical factor. We only know we must use the same root of unity in both. We have no means of deciding on a root of unity by algebraic resolution unless we descend into arithmetic. We must consider the characters of arbitrary symbols. If we choose that the $a_i \in \mathbf{R}$, we could determine the cases where one or more x_i are real and find our root of unity. But algebraic resolution requires our coeffs and roots to remain arbitrary symbols without attributes or meaning.

Consider our quad in y where equality of dscrs makes it a resolvent of its cubic in x. (12) appears only as $(12)^3$. So the quad is the same if expressed as

$$(x - a_1)^3 - 3(12)(x - a_1) - (13) = 0 \qquad [1]$$

or

$$(x - a_1)^3 - 3\omega(12)(x - a_1) - (13) = 0 \qquad [2]$$

or

$$(x - a_1)^3 - 3\omega^2(12)(x - a_1) - (13) = 0 \qquad [3]$$

which however have no root in common. And here we lose all trace of which of these we are solving. Our solns are the same, no matter which of this trio we are using.

But consider that the intersubstitution of $(x - a_1)$, $\omega(x - a_1)$, and $\omega^2(x - a_1)$ changes any of the three into one of the others. So the **form** of the root must allow this substitution. In itself, y is not capable of this. There must be some fn of y that allows it. So we are led to consider three systems which solve our three cubics:

$$y_1^{1/3} = A((x_1 - a_1) + \omega(x_2 - a_1) + \omega^2(x_3 - a_1))$$
$$y_2^{1/3} = A((x_1 - a_1) + \omega^2(x_2 - a_1) + \omega(x_3 - a_1))$$

for [1]

$$y_1^{1/3} = A(\omega(x_1 - a_1) + \omega^2(x_2 - a_1) + (x_3 - a_1))$$
$$y_2^{1/3} = A(\omega(x_1 - a_1) + (x_2 - a_1) + \omega^2(x_3 - a_1))$$

for [2] and

$$y_1^{1/3} = A(\omega^2(x_1 - a_1) + (x_2 - a_1) + \omega(x_3 - a_1))$$
$$y_2^{1/3} = A(\omega^2(x_1 - a_1) + \omega(x_2 - a_1) + (x_3 - a_1))$$

for [3]. The ambiguity which remains can be resolved if we adhere to considering x_i as symbols only. If we had known of this ambiguity of the roots of unity, we would have known that any quad resolvent of a cubic must have the dscr of the cubic. If $y_1^{1/3}$ and $y_2^{1/3}$ must be expressible as rational linear fns of x_i, i.e. of $(x_i - a_1)$, then so must be y_1 and y_2. And our

$$y^2 - 2N_1(13)y + N_2(12)^3 = 0$$

will work here only if

$$N_1{}^2(13)^2 - N_2(12)^3$$

is a perfect square which gives us our equal dscrs. So N_1 and N_2 are bound by this condition. Conversely, if the dscrs are equal, the roots of the quad have the qualities that make it a resolvent, not only for its cubic, but for the conjugate cubics which in resolution cannot be separated from it.

That our quad qfn with a dscr equal to that of a cubic is a resolvent is not apodictically established. It appears as the result of a calculation. This identity of dscrs has the following aspects: y is of 3° in the roots x_i; coeffs of the quad are sym. fns of these roots of an appropriate degree; when two x_i are equal, the y are equal so that the whole of the irrational part of the quad roots appears in the cubic's roots; and when all x_i are equal, the eqn in y vanishes along with the irrational part of the expression for x.

Conversely, as S_2 and S_6 are fns of $(x_i - a_1)$ [1-3], the irrational part of the cubic's roots are fns of y_1 and y_2 exclusively, where their irrational part occurs in them unaltered. Because we can't distinguish between y_1 and y_2, they must enter into these fns symmetrically. So the x's are sym. irrational fns of y_1 and y_2 exclusively, fns of one y combined with the same irrational fn of the other.

This fn of y is unaltered by the intersubstitution of $(x - a_1)$, $\omega(x - a_1)$, and $\omega^2(x - a_1)$. At the same time, there must be some f(y): when $(x - a_1)$ becomes $\omega(x - a_1)$, $f(\omega(x - a_1)) = \omega f(x - a_1)$ and Sym. for ω^2 OR $y = (y^{1/3})^3 = (\omega y^{1/3})^3 = (\omega^2 y^{1/3})^3$ OR y is a perfect cube of some three-valued fn of the $(x - a_1)$'s. Therefore, y is rational. This is seen in the form of y as $S_3 \pm \sqrt{S_6}$ where S_3,S_6 are fns only of the three values of $(x - a_1)^3$. From algebraic considerations only where x_i,a_i are mere symbols, our soln of the cubic takes the form

$$S_1 + \sqrt[3]{(S_3 + \sqrt{S_6})} + \sqrt[3]{(S_3 - \sqrt{S_6})}$$

which is a fn of x_i [1-3] and is nine-valued and solves three distinct cubics, solns differing by a cubic root of unity.

We see from the identity of the critical fns of both the cubic and quad that y must be rational and a two-valued fn, each value a perfect cube of a linear fn. And the resolvent in y resolves three related cubics. Because (12) is in the quad only as a cube, the quad stands in the same relation to all three cubics, whose roots are different. And here again, y must be rational and the six values of $y^{1/3}$ must be linear in the x's.

Besides the lin.trans., only one other transformation maintains this form. This is the Tschirnhausen transformation (T-trans) where (21) vanishes. This reduces the cubic to a binomial form and gives the same form to the roots as above. Therefore the soln of the cubic is unique in form.

We can view the cubic as we did the quadratic, as a fn of t. The cubic has form

$$(x - a_1)^3 - 3(12)(x - a_1) - (13) = 0$$

Let $(x - a_1) = \sqrt{(3(12))} \cdot t$ and the form is

$$t^3 - t = (13)/(3(12))^{3/2} = (\text{say})\ v$$

Or $\varphi t = v$ and our cubic soln is φ^{-1} in $t = \varphi^{-1}v$

The Quartic (First Method)

We infer from the lin.trans., that if the quartic is resolved by a lin.trans., the rational part of the roots is a_1 and the irrational parts are fns of (12),(13),(14). We look for a cubic with the same dscr (to a numerical factor) as the quartic. And we expect this cubic to be a resolvent whose roots relate to the roots of the quartic.

The dscr of a quartic is 12° in its roots and a cubic's is 6°. The argument of a cubic in y with the same dscr as a cubic in x must be 2° with coeffs of degrees 0,2,4,6. The most complete form of this is

$$y^3 - 3N_1(12)y^2 + 3(N_2(12)^2 + N'_2(14))y - (N_3(12)^2(14) + N'_3(13)^2) = 0$$

Given these constants, or rather, their ratios, we want to give them values so that this cubic's dscr is that of the quartic. The dscr of this cubic is

$$4(\text{I·II})^3 - (\text{I·III})^2$$

where

$(\text{I·II}) = (N_1^2 - N_2)(12)^2 - N'_2(14)$ OR $C(12)^2 - N'_2(14)$
$(\text{I·III}) = (2N_1^3 - 3N_1N_2)(12)^3 - (3N_1N'_2 - N_3)(12)(14) + N'_3(13)^2$
OR $D(12)^3 - E(12)(14) + N'_3(13)^2$

and the dscr of the quartic is given in our earlier table. Comparing the two

	Dscr Cubic	**Dscr Quartic**
$(12)^2$	$4C^3 - D^2$	0
$(12)^4(14)$	$2DE - 12C^2N'_2$	81
$(12)^2(14)^2$	$12CN'^2_2 - E^2$	18
$(14)^3$	$-4N'^3_2$	1
$(12)^3(13)^2$	$-2DN'_3$	-54
$(12)(13)^2(14)$	$2EN'_3$	-54
$(13)^4$	$-N'^2_3$	27

From this we have

$$4C^3 - D^2 = 0$$
$$108N'^2 - N'_3{}^2 = 0$$
$$DN'_3 + 108N'_2{}^3 = 0$$
$$EN'_3 - 108N'_2{}^3 = 0$$
$$6C^2N'_2 - DE - 162N'_2{}^3 = 0$$
$$12CN'_2{}^2 - E^2 + 72N'_2{}^3 = 0$$

The first implies

$$N_2 = 3/4 \cdot N_1{}^2 \quad \text{OR} \quad C = ¼N_1{}^2$$

∴

$$N'_2 = ⅓C = 1/12 \cdot N_1{}^2$$
$$N'_3 = (108N'_2{}^3)^{1/2} = ¼N_1{}^3$$
$$E = 3N_1N'_2 - N_3 = ¼N_1{}^3 - N_3$$

But E must equal N_3 or $¼N_1{}^3$ ∴ $N_3 = 0$

We have to satisfy these six eqns above with only four constants. But this is possible, no matter what value is given to N_1. Let $N_1 = 6$ to avoid fractions and the cubic in y is

$$y^3 - 18(12)y^2 + 3(27(12)^2 + 3(14))y - 54(13)^2 = 0$$

We can resolve this cubic expressing its three roots in y in terms of (12),(13),(14), which is to say in terms of a_i [1-4] and also in terms of the x's under radicals by subbing for each coeff its corresponding value in x's. This form is

$$y = S_2 + \sqrt[3]{(S_6 + \sqrt{S_{12}})} + \sqrt[3]{(S_6 - \sqrt{S_{12}})}$$
$$= S_2 + \sqrt[3]{(S_6 + R_6)} + \sqrt[3]{(S_6 - R_6)}$$

where $\sqrt{S_{12}}$ is the square root of the quartic expressed in terms of its roots and is therefore reducible. At this point, we could expand $S_6 \pm R_6$ in terms of the x's and take cube roots. Each value in y would be rational and a perfect square producing a linear fn of the x's. But this is computation and we seek a general theory.

If we assume the coeffs of our cubic to be rational, all we need flows from the form of its having the same dscr as the quartic. Even without this assumption, there must be at least one common radical in the expression of the roots and this is a step toward showing $y \in \mathbf{Q}$ as S_{12} is a perfect square wrt the x's.

When the cubic has three equal roots, the dscr vanished and so do (I·II),(I·III) and we get

$$(14) - 3(12)^2 = 0$$
$$(12)^3 + (12)(14) - (13)^2 = 0$$
$$\therefore \quad 4(12)^3 - (13)^2 = 0$$

So this last, with vanishing dscr, gives us three equal roots in the quartic.

All conditions between coeffs which show multiplicity of roots in the cubic operate in a similar manner in the quartic. So **all** the radicals in the cubic's roots must be unaltered in the quartic's roots. The is the property, or definition, of a **resolvent**.

If we take

$$R_2 + \omega R'_2 + \omega^2 R''_2 \qquad\qquad R_2 + \omega^2 R'_2 + \omega R''_2$$

and multiply each by ω and ω^2, the conditions of three equal roots in a quartic are represented by the product of these six fns where the R's are 2° qfns of the quartic's roots. Indirectly, this shows the y's are rational. But we need to establish this directly from the relation of the cubic in y to the quartic in x and the fact that the product of the cubic's roots is the square of a qfn of the quartic's coeffs.

From the dscrs, we know that y_i [1-3] enter into the four $(x - a_1)$'s, preserving their irrational parts. So these $(x - a_1)$'s are sym. wrt y_i and must be irrational fns of them, consisting of irrational fns of each y_i separately or (same thing) combined in pairs. And this product of the cubic roots is allied to another quartic in the same manner. This connection, being identical, we can't tell which quartic our cubic comes from by working backwards. So the cubic solves two quartics with distinct roots. In our cubic, (13) appears only as $(13)^2$ and can come from either of these:

$$(x - a_1)^4 - 6(12)(x - a_1)^2 \mp 4(13)(x - a_1) - (14) = 0$$

Therefore, y is a fn of the four values of $(x - a_1)$ only through the four values of $(x - a_1)^2$. So y as a fn of $(x - a_1)$ cannot undergo any change when $(x - a_1)$ is changed into $-(x - a_1)$. The root of the quartic must be expressed through some f(y) that admits of a change of value in this substitution as the cubic must solve two quartics. So there must be some fn ψ operating only on y which changes when $(x - a_1)$ becomes $-(x - a_1)$. But this change is obliterated when we use ψ^{-1} on ψ(y). So ψ^{-1} is squaring and ψ is taking the square root. And this requires y to be rational in terms of the x's. For if the cube roots were irreducible surds, any root of a power or fn of y would also have that form and would contain y only in the four values of $(x - a_1)^2$. Because the three values of y are rational and because their product is a perfect square then each of the y's is a perfect square and a linear fn of the four values of $(x - a_1)$. And for the two quartics, one set of y's is the negative of the other.

Now it need not be necessary that three quantities in **Q** are a square because their product is one. They could have the form RR', RR", R'R". And then $(13)^2$ would have three factors, each repeated twice. But every factor, as we have just shown, can differ by sign only. And with symmetry of roots of unity, we must have (using a_1 for convenience only):

$$\sqrt{y_1} = \pm A((x_1 - a_1) + (x_2 - a_1) + (x_3 - a_1) + (x_4 - a_1))$$
$$\sqrt{y_2} = \pm A((x_1 - a_1) - (x_2 - a_1) + (x_3 - a_1) - (x_4 - a_1))$$
$$\sqrt{y_3} = \pm A((x_1 - a_1) - (x_2 - a_1) - (x_3 - a_1) + (x_4 - a_1))$$

and these solve:

$$(x - a_1)^4 - 6(12)(x - a_1)^2 - 4(13)(x - a_1) - (14) = 0$$
$$(x - a_1)^4 - 6(12)(x - a_1)^2 + 4(13)(x - a_1) - (14) = 0$$

In computation, we must use the square root in one case so that $y_1y_2y_3$ has the same sign as (13) and in the other case, as -(13). The fn which expresses our solution by this method is

$$S_1 + \sqrt{(S_2 + \sqrt[3]{(S_6 + \sqrt{S_{12}})} + \sqrt[3]{(S_6 - \sqrt{S_{12}})})}$$
$$+ \sqrt{(S_2 + \omega\cdot\sqrt[3]{(S_6 + \sqrt{S_{12}})} + \omega^2\cdot\sqrt[3]{(S_6 - \sqrt{S_{12}})})}$$
$$+ \sqrt{(S_2 + \omega^2\cdot\sqrt[3]{(S_6 + \sqrt{S_{12}})} + \omega\cdot\sqrt[3]{(S_6 - \sqrt{S_{12}})})}$$

Here we are unable to express the quartic problem in the form $\varphi t = v$. That multiple of $(x - a_1)$ we call t is not in our expression of the root a fn of any one symbol v. So at this point, the root of a general quartic cannot be represented as a fn with a single parameter. But we will return to this idea in the next section.

The Quartic (Second Method)

As a recap, for our *a priori* soln of cubic and quartic we have:

1) For each, we have an eqn of the next inferior degree whose coeffs are qfns of the given eqn's coeffs and which has the same dscr as the given eqn.

2) The constant term of this lesser eqn, the product of its roots, is a perfect power of a qfn of the given eqn's coeffs, the exponent of the power being the degree of the lesser eqn's argument.

3) This qfn of the constant enters the lesser eqn only in that power of these arguments.

We also know that our process solves a set of eqns with distinct roots. The set has as many elements as the power of the constant term in y. Both dscrs take the same form: $S^2 + NS'^3$. Our y must be not only rational in its x's but a power of a linear qfn of the x's.

The cubic, with its resultant, presented no problems. In the quartic, having to satisfy six eqns with four ratios prevented us from knowing *a priori* that we could so satisfy them. So we could not know that an *a priori* soln existed. This is important and at least shows us that our soln was not the simplest and most natural.

A remedy would be to reduce the number of eqns to satisfy. To equate two general 12° fns of (12),(13),(14), we must satisfy six eqns, there being seven terms with coeffs. In our comparison of cubic and quartic, if we strike out (12), we have two terms left and the ratio of their coeffs can be determined without redundant eqns of other inconsistency.

A quartic with (12) = 0 has this form

$$3(\sum z)^2 - 8\sum(zz) = 0$$

and has the conditional form

$$z^4 - 4b_1z^3 + 6b_1{}^2z^2 - 4b_3z + b_4 = 0$$

and has the same dscr as

$$y^3 + 9(41)y - 51(31)^2 = 0$$

Resolving this, we get three values of y in terms of (41),(31) or b_1,b_3,b_4. But we cannot express y uniquely in terms of z. We could sub $\frac{1}{4}\sum z \to b_1$, $\frac{1}{4}\sum(zzz) \to b_3$, and $z_1z_2z_3z_4 \to b_4$ but the condition $3(\sum z)^2 = 8\sum(zz)$ admits infinite variation. So we need a unique expression of y. And even then, we have no resolution of the general quartic in x.

Let's first consider transforming any quartic into our (12) = 0 quartic for which we use the T-trans. We are given

$$x^4 - 4a_1x^3 + 6a_2x^2 - 4a_3x + a_4 = 0$$

and we need to transform it into a quartic in z where z is some ψx that makes (21) = 0. (At this point, note that if the (21) seems a typo or you are unsure of its meaning, you are not quite yet following the notation. I say this as I had the same issue.) The general form of our ψ is

$$z = \psi x = hx^3 + kx^2 + lx$$

any higher power of x being reducible and any constant term absorbed by z. The transformed eqn in z is therefore

$$z^4 - \sum(\psi x)z^3 + \sum(\psi x\psi x)z^2 - \sum(\psi x\psi x\psi x)z + \prod\psi x_i \; [1\text{-}4]$$

and for (21) = 0, we need h,k,l:

$$3(\sum(\psi x))^2 - 8(\sum(\psi x\psi x)) = 0$$

or in our, now almost forgotten, underscore notation of sym. fns of the roots, this is:

$$\underline{33}h^2 + 2\underline{32}hk + 2\underline{31}hl + \underline{22}k^2 + 2\underline{21}kl + \underline{11}l^2 = 0$$

We can satisfy this in two ways with three or two results. With three constants and two conditions we can make h = 0 and determine k:l by

$$\underline{22}k^2 + 2\underline{21}kl + \underline{11}l^2 = 0$$

and get two forms of ψ each with a quartic of form (12) = 0. This gives a set of two quartics and Sym. for k = 0 and l = 0. Secondly, our condition takes form

$$(\underline{33}h + \underline{32}k + \underline{31}l)^2 + (\underline{33}\cdot\underline{22} - \underline{32}\cdot\underline{32})k^2 + 2(\underline{33}\cdot\underline{21} - \underline{32}\cdot\underline{31})kl + (\underline{33}\cdot\underline{11} - \underline{31}\cdot\underline{31})l^2 = 0$$

which is equivalent to two linear conditions

$$\underline{33}h + \underline{32}k + \underline{31}l = 0$$
$$Ak + (B + \sqrt{(B^2 - AC)})l = 0$$

where A,B,C are the three sym. fns above of degrees 10°,9°,8° respectively. Any convenient h gives k and l by a linear process and again z takes two forms, giving again two quartics.

The reduction can also be made by exchanging 1 and 3, h and l in the above. In all cases, a distinct form of irreducible irrational appears for each method of elimination. The degrees of the new fns introduced are 14° and 18°. All methods appear to be distinct. In every case, we get two quartics differing by the sign of the $^2\sqrt{}$ which is in every coeff. We consider now only the simplest of the above transforms where z is 2° in x:

$$z = kx^2 + lx = \underline{11}x^2 + (\underline{21} \pm \sqrt{(\underline{21}\cdot\underline{21} \pm \underline{22}\cdot\underline{11})})x$$

OR

$${}^1/_{16} \cdot z = 3(a_1^2 - a_2)x^2 + (3(4a_1^3 - 5a_1a_2 + a_3) \pm \sqrt{(\, 9(4a_1^3 - 5a_1a_2 + a_3)^2}$$
$$- 3(a_1^2 - a_2)(48a_4 - 72a_1^2a_2 + 9a_2^2 + 16a_1a_3 - a_4)\,)x$$

The radical last written down is the essential irrational form which will pervade the whole of our future results. It is irreducible or, at least, we will treat it as such regardless. With this z, $b_1 = ¼\sum z$, $b_3 = ¼\sum(zzz)$, and $b_4 = \prod z_i$ [1-4] -- all expressible in terms of a_i [1-4], all containing this last radical but otherwise rational.

If this leads to a result, the irrationals in it will **not** be fns of (12),(13),(14); the rational part with **not** be a_1; and the sum of the roots **will** be $4a_1$. Our pair of conditioned quartics with then be

$$z^4 - 4b_1z^3 + 6b_1^2z^2 - 4b_3z + b_4 = 0$$

where $z = kx^2 + lx$ and where l:k, b_1, b_3, b_4 are 2° irrational fns of a_i [1-4] each being two-valued, one for each quartic in the set according to the square roots of unity. Consider this cubic of one of our conditioned quartics:

$$y^3 + 9(41)y - 54(31)^2 = 0$$

We can express (31) as $b_3 - b_1^3$ and (41) as $-(b_4 - b_1^4) + 4b_1(b_3 - b_1^3)$ which is all in terms of a_i [1-4] and each contains our big quadratic irrational (quad-irrational) which we denote $\sqrt{I_6}$. So we can express y completely in terms of a_i [1-4] and therefore by sym. fns of the x's or roots.

In both cases, the irrational is of Hamilton's 3d order. And in both cases, in whatever mode of transformation, the y's are now uniquely expressed.

Consider the relation of the quartic in z (ψx) and the cubic in y. We ask if the latter is the resolvent of the former. Both have the same dscr. So any expressible radical in the y's will enter the root of the quartic which is

$$(z - b_1)^4 - 4(31)(z - b_1) - (41) = 0$$

With these two parameters, we cannot impose two conditions nor suppose three or four equal roots without the cubic vanishing and the quartic being $(z - b_1)^4 = 0$, So any multiplicity in y is reflected in z (ψx) and radicals in y enter the roots of the quartic unaltered. These radicals are the three forms of

$$\sqrt[3]{((31)^2 + \sqrt{((31)^4 + (⅓(41))^3)}) + \sqrt{((31)^2 - \sqrt{((31)^4 + (⅓(41))^3)})}}$$

and are uniquely expressible in x's or a's.

Although all this is not definitely expressible in z, we can definitely express the dscr, if not uniquely, and it is a perfect square, which suffices for our theory. Properly speaking, there is no fn of z as z is identically ψx which is $kx^2 + lx$. We do not attempt to solve a conditioned quartic but only the complete quartic in x which takes form

$$(\psi x - b_1)^4 - 4(31)(\psi x - b_1) - (41) = 0$$

When the cubic has three equal roots, it is $y^3 = 0$ and (31),(41) vanish and both radicals vanish from the roots. Then the quartic becomes

$$(z - b_1)^4 = (\psi x - b_1)^4 = 0$$

with three equal roots, or considered in z, 4 equal roots. In x, three equal roots reduces the quartic to $\psi x - b_1$ and with four equal roots, $x = a_1$. Our unchanging radical of the cubic's and quartic's roots must be a fn of y_1 and Sym. the same of y_2 and y_3. The quad-irrational (31) appears only as $(31)^2$ in the cubic of y. So -(31) gives us the conjugate quartic's resolvent and our original quartic being

$$(z - b_1)4 - 4(31)(z - b_1) - (41) = 0$$

its conjugate is

$$(z - b_1)^4 + 4(31)(z - b_1) - (41) = 0$$

where we sub $-(kx^2 + lx - b_1)$ for $(kx^2 + lx - b_1)$. So y remains the same but some fn of y must change in this substitution. Again we have $y^{1/2}$ changing but $(y^{1/2})^2$ does not, as in the first method. Here y, in terms of the x's, must be a qfn, not of the x's, but of the expressions of z in terms of x. So y must be a linear fn of ψx_i [1-4].

Then, except for our irreducible $^2\sqrt{}$ radical, y is reducible to a 2° qfn of ψx_i [1-4] or

$$A_1(\psi x_1 - b_1) + A_2(\psi x_2 - b_1) + A_3(\psi x_3 - b_1) + A_4(\psi x_4 - b_1) = y_1^{1/2}$$
$$B_1(\psi x_1 - b_1) + B_2(\psi x_2 - b_1) + B_3(\psi x_3 - b_1) + B_4(\psi x_4 - b_1) = y_2^{1/2}$$
$$C_1(\psi x_1 - b_1) + C_2(\psi x_2 - b_1) + C_3(\psi x_3 - b_1) + C_4(\psi x_4 - b_1) = y_3^{1/2}$$

which combined with

$$(\psi x_1 - b_1) + (\psi x_2 - b_1) + (\psi x_3 - b_1) + (\psi x_4 - b_1) = 0$$

determine ψx_i [1-4]. The A_i, B_i, and C_i are numerical constants which arise in the computation. The y_i [1-3] are the previously determined irrational fns of a_i [1-4]. Because we have a set of two quartics, the above process can be repeated to obtain a second set of solns ψx_i [1-4]. We first determine

$$kx_1^2 + lx_1 \qquad kx_2^2 + lx_2 \qquad kx_3^2 + lx_3 \qquad kx_4^2 + lx_4$$

where l is $\underline{21} + \sqrt{(\underline{21}\cdot\underline{21} - \underline{22}\cdot\underline{11})}$ then we determine $kx_i^2 + l'x_i$ where l' is $\underline{21} - \sqrt{(\underline{21}\cdot\underline{21} - \underline{22}\cdot\underline{11})}$. Any sym. fn of $kx_1^2 + lx_1$ and $kx_1^2 + l'x_1$ is a fn of x_1 whose coeffs are qfns of a_i [1-4]. From this, by eliminating all powers of x_1 above the first, we absolutely determine x_1.

Example

Let our sym. fn be the sum of these which is

$$\underline{11}x_1^2 + \underline{21}x_1 = V_a$$

where V_a is a known fn of a_i [1-4]. Then we eliminate x_1^4, x_1^3, and x_1^2 from these two fns:

$$x_1^4 - 4a_1x_1^3 + 6a_2x_1^2 - 4a_3x_1 + a_4 = 0$$
$$(a_1^2 - a_2)x_1^2 + (4a_1^3 - 5a_1a_2 + a_3)x_1 - V_a = 0$$

Here, to extract the roots, we need only determine the double set of $\sqrt{y_i}$ [1-3]. And the irrational fns of a_i [1-4] in one set appear in the other as our $\sqrt{I_6}$ with the sign changed.

Our second method here is a **new** form of quartic soln where each term of it contains only two elements. If our ψ had been $hx_3 + lx$, the soln would take another form with the irreducible radical $\sqrt{I_8}$; if with $hx^3 + kx^2$, a new form with $\sqrt{I_{10}}$; if with the entire $hx_3 + kx^2 + lx$, yet another form with one of the other irreducible radicals of degrees $\sqrt{I_{14}}$ or $\sqrt{I_{18}}$.

The first method's results could not have been anticipated *a priori* : that certain qfns of the roots of the quartic were also roots of the cubic with coeffs of qfns of the quartic's coeffs. This meant that there are qfns of four symbols with exactly three values.

With this second method, we have combinations, not of coeffs, but of a certain irrationality of the form A + B√I where A,B∈**Q**, I∈**Q** but not necessarily a perfect square. There are several distinct forms of the √I, each with its mode of soln. From this method's fns of the roots containing our √I, we again find a three-valued fn of four symbols, solving the quartic.

The two methods of soln are independent. If we had ignored the first method, we would have no knowledge of the existence of a three-values qfn of four symbols. The two methods show that any eqn insoluble by a process where the rational part of the roots is a_1 and the irrational parts are fns of (12), (13), ... may be soluble by the introduction of some irrational fn, an appropriate analog of our √I.

In considering our second method, it is significant that our A_i,B_i,C_i were discoverable through operations only without reference to the first method. Comparing the two methods and knowing the eqn in z is a case of the general quartic, we find the A_i,B_i,C_i the same in each. In the second process, each root is expressed in terms of six irrational fns of the coeffs, each of Hamilton's 4th order, besides the quasi-rational part which is the sum of two values of b_i. Each root can be expressed less symmetrically by fewer fns. We have **not** been solving the general quartic by means of the reduced, conditioned, trinomial quartic. We have taken one of the solns of the general quartic and imparted into it this limitation or condition.

The second method **does** involve one or the other of five quad-irrationals introduced by the T-trans. No others would achieve a soln distinct from the first method. But using this transformation, we lose, from the soln stated at the end of the first method, the term S_2. In that statement of soln, S_2 could only vanish if we had some relation between coeffs and roots. In our search for generality, we admit no such relation of condition. But here, by subbing some χx for x in the quartic, some one of S_2, S_6, S_{12} may vanish without affecting generality or introducing a relation or condition. The T-trans affects S_2.

Note that this transform, having removed S_2 by a process acting on the radicals, cannot be undone by a process outside the radicals. Let the root be ρ. If we sub χx for x in ρ, the result will be equivalent to, and reducible to, χρ so long as the form of ρ is not altered by the substitution. But if χ produces alterations of form, the result cannot take the form χρ although it be arithmetically equivalent. This vanishing of S_2 is of great theoretical importance. Algebraically, it presents the resolution in a different form. Here, like the quad and cubic, the quartic is a fn of a single parameter -- $(31)^4/(41)^3$ -- which **cannot** be inferred from the first method.

We can introduce any lin.trans. into the original quartic before applying the T-trans. And each lin.trans. changes our √I. These are not distinct results but merely linear variations of the soln. But if we begin with the lin.trans. that causes the second term of the quartic to vanish, we greatly simplify our work without loss of generality of introduction of any condition. We make this our standard mode of transformation. It is effected by subbing x - a_1 for x, making a_1 zero in all formulae, and then a_2,a_3,a_4 are -(12),(13),-(14). To show we are doing this, we denote a_2,a_3,a_4 as c_2,c_3,c_4.

We now have

$$\psi x = k(x - a_1)^2 + l(x - a_1)$$

where $k = -3c^2$ and $l = -3c_3 + \surd(3(9c_2^3 - c_2c_4 + 3c_3^2))$. Then $b_1 = 9c_2^2$ and

$$4b_3 = l^3\sum(xxx) + l^2k\sum(x^2xx) + lk^2\sum(x^2x^2x) + k^3\sum(x^2x^2x^2)$$

or

$$b_3 = c_3l^3 + 3c_2c_4l^2 + 54c_2^3c_3l - 27c_2^3(4c_3^2 - 3c_2c_1)$$

which divided by

$$l^2 + 6c_3l - 3c_2(9c_2^2 - c_4) = 0$$

gives

$$b_3 = 81c_2^4 - 513c_2^3c_3^2 - 9c_2^2c_4^2 + 162c_2c_3^2c_4 - 108c_3^4 + (81c_2^3c^3 - 21c_2c_3c_4 + 36c_3^2)\surd I_6$$

Sym.

$$b_4 = c_4(l^4 + 54c_2^3l^2 - 108\, c_2^3c_3l + 81c_2^4c_4)$$
$$= 9c_4(243c_2^6 - 36c_2^4c_4 + 324c_2^3c_3^2 + c_2^2c_4^2 - 24c_2c_3^2c_4 + 72c_3^4 - (72c_2^3c_3 - 4c_2c_3c_4 + 24c_3^3)\surd I_6)$$

These give

$$(31) = -729c_2^6 + 81c_2^4c_4 - 513c_2^3c_3^2 -9c_2^2c_4^2 + 162c_2c_3^2c_4 - 108c_3 + (81c_2^3c_3 - 21c_2c_3c_4 + 36c_2^3)\surd I_6$$
$$= \text{(for reference)} \quad B_{12} + B_9\surd I_6$$

$$⅓(41) = -6561c_2^8 + 243c_2^8c_4 - 6156c_2^5c_3^2 + 972c_2^3c_3^2c_4 - 1296c_2^2c_3^4 - 3c_2^3c_4^3 + 72c_2c_3^2c_4^2 - 216c_3^4c_4 + 12(81c_2^5c_3 - 3c_2^3c_3c_4 + 36c_2^2c_3^3 - c_2c_3c_4^2 + 6c_3^3c_4)\surd I_6$$
$$= \text{(for reference)} \quad B_{16} + B_{33}\surd I_6$$

from which we form

$$y = 3(\sqrt[3]{}((31)^2 + \surd((31)^4 + (⅓(41))^3)) + \sqrt[3]{}((31)^2 + \surd((31)^4 + (⅓(41))^3)))$$

where $(31)^2 = B_{12}^2 + B_9I_6 + 2B_9B_{12}\surd I_6$ and $(31)^4 + (⅓(41))^3 =$

$$B_{12}^4 + 6B_{12}^2B_9^2I_6 + B_9^4I_6 + B_{16}^3 + 3B_{16}B_{13}^2I_6 + (4B_{12}^3B_9 + 4B_{12}B_9^3I_6 + 3B_{16}^2B_{13} + B_{18}^3I_6)\surd I$$

Suppose the quartic to have two equal roots so that y has these values

$$6(31)^{2/3} \quad -3(31)^{2/3} \quad -3(31)^{2/3}$$

$$\therefore \quad z_1 + z_2 - z_3 - z_4 = 4(31)^{1/3}$$
$$z_1 + z_2 + z_3 - z_4 = 2\sqrt{-2}\,(31)^{1/3}$$
$$z_1 - z_2 - z_3 + z_4 = 2\sqrt{-2}\,(31)^{1/3}$$
$$z_1 + z_2 + z_3 + z_4 = 36c_2^2$$

$$\therefore \quad z_1 = 9c_2^2 + \sqrt[3]{(31)} \cdot (1 + \sqrt{-2})$$
$$z_2 = 9c_2^2 + \sqrt[3]{(31)} \cdot (1 - \sqrt{-2})$$
$$z_3 = 9c_2^2 - \sqrt[3]{(31)}$$
$$z_4 = 9c_2^2 - \sqrt[3]{(31)}$$

where $z = -3c_2(x - a_1)^2 - (3c_3 - \sqrt{I_6})(x - a_1)$

If in (31) we change the sign of I_6, we have Sym. values for

$$-3c_2(x - a_1)^2 - (3c_3 + \sqrt{I_6})(x - a_1)$$

and summing these gives

$$-6c_2(x - a_1)^2 - 6c_3(x - a_1) \equiv \psi x$$

Let $R = \sqrt[3]{(B_{12} + B_9\sqrt{I_6})} + \sqrt[3]{(B_{12} - B_9\sqrt{I_6})}$ and for our ψx we have

$$\psi x_1 = 18c_2^2 + (1 + \sqrt{-2})R$$
$$\psi x_2 = 18c_2^2 + (1 - \sqrt{-2})R$$
$$\psi x_3 = 18c_2^2 - R$$
$$\psi x_4 = 18c_2^2 - R$$

The ordinary method would give equivalent arithmetical results but these cube root radicals would not appear at all and the expression would have no surds. But (31) shows its cube root cannot be algebraically taken as it contains no symbol of the first degree.

Now if I_6 equalled zero, we would arrive at the first method's soln by the second method. The dscr of the quartic in z (ψx) contains the dscr of the quartic in x as a factor. When the latter is zero, so is the former. But not conversely. When the former is zero, either the quartic in x has two equal roots OR for some two values of the x's (say x_1, x_2) that $k(x_1 + x_2) + l = 0$. We have four cases:

1) eqn in x has two equal roots ⇒ dscrs of both conditioned quartics vanish
2) $k(x_1 + x_2) + l = 0$ ⇒ 1st conditioned quartic's dscr vanishes, 2d's does not
3) $k(x_1 + x_2) + l' = 0$ ⇒ 2d conditioned quartic's dscr vanishes, 1st's does not
4) $I_6 = 0$ ⇒ Neither dscr vanishes but the two conditioned quartics become only one and z is an ifn of x: $3(a_1^2 - a_2)x^2 - 3(4a_1^3 - 5a_1a_2 + a_3)x$ and both methods yield identical solns.

In the second method, we operate on a resolvent cubic but not the resolvent of any other method. It has been shown that the resolvents in Euler's and Simpson's methods are essentially the same. Our new method, instead of a resolvent cubic, actually has a resolvent sextic. And this is a qfn divisible into two cubics with quad-irrational coeffs. Finally, we see the soln of a quartic **is** an ordinary algebraic inverse operation. The general quartic has form

$$(z - b_1)^4 - 4(31)(z - b_1) - (41) = 0$$

make $z - b_1 = \sqrt[3]{(4(31)} \cdot t$

$$\therefore \qquad t^4 - t = (41) / \sqrt[4/3]{(4(31))} = v$$

or $\varphi t = v$ and our resolution is $\varphi^{-1} v = t$.

The T-trans is also used in the Tschirnhausen Solution of the quartic where a quartic in x becomes a quartic in t under the condition (13) = 0 instead of our (12) = 0. This is done by means of a cubic and the eqn in t is a quartic with only even powers of t and solved by taking square roots of a quadratic whose coeffs in terms of a_i [1-4] come from the cubic. This leads to the same cubic resolvent as other methods. So our second method is the only one distinct from all the other linear ones equivalent to the first method.

The Quintic

If we could express the root of a quintic by the first method of lin.trans., the rational part of it would be a_1 and the irrational part fns of (12),(13),(14),(15). We would need a quartic whose coeffs are qfns of those four values with a dscr equal or a numeric multiple of the quintic's. And its roots would be a fn of the five $(x - a_1)$'s. Let us see why this won't work.

In their roots, the dscr of a quintic is 20° and of a quartic, 12°. So no degree of argument in a quartic will work. But lcm(12,20) = 60 and if the argument of the quartic is 5°, its dscr is then 60° and equal to the cube of the quintic's. Its coeffs will be of degrees 0, 5, 10, 15, and 20. We could algebrate all of this the hard way. But there is a valid, much lazier approach. We need the principal critical fns of both quartic and quintic to share the condition of two equal roots. When the first derivative of the quantic has three equal roots, its dscr is a perfect cube and the quartic has form

$$(y - l)^4 + m = 0$$

where the dscr is the cube of m. Recall the honking big dscr of the quintic from our earlier table and lazily denote it Δ. Let $m = N_3\Delta$. Then the most general form of l is

$$N_1(12)(130 + N_2(15)$$

and our quartic becomes

$$(y - (N_1(12)(13) + N_2(15))^4 - N_3\Delta = 0$$

Solving for y, its expression in terms of x's is irreducible. Its roots contain $\sqrt[4]{\Delta}$ which is $\sqrt{(\sqrt{\Delta})}$. As Δ is a perfect square in the x's, the first $\sqrt{}$ works just fine. But the square root of Δ has the form $\sum xx^2x^3x^4$ with half the terms negative and so is not a perfect square. Our y here is **not** any fn of the $(x - a_1)$'s. If $\sqrt[4]{\Delta}$ were reducible, we would have a five-symboled fn possessing four values and that is algebraically impossible. So we can see that our first method of lin.trans. cannot work.

So we need to proceed, if possible, by our second method, introducing irreducible irrationals and avoiding lin.trans. We cannot find a four-valued qfn of five symbols. But we may find irrational fns of known, specific irrationality of five symbols whose multiplicity of values may solve the quintic. Note that although the y's are irrational, they divide into two pairs whose products are rational wrt the roots:

$$y_1y_2 = (N_1(12)(13) + N_2(15))^2 - \sqrt{(N_3\Delta)}$$
$$y_3y_4 = (N_1(12)(13) + N_2(15))^2 + \sqrt{(N_3\Delta)}$$

And this relation should be kept in mind as we go forward. Let's first digress by asking if there are conditioned quintics with $\sqrt[4]{\Delta}$ reducible. If Δ in terms of coeffs is a square fn, then it **could** be a fourth power of the roots. If this occurs, our two products above would not only be rational but would be sym. fns of the roots. So if in our tabular Δ we make (13) = 0 and condition (12) *vis a vis* (14), we can make Δ a perfect square where relations between coeffs have a simple character. Let (13) = 0 and then let $(14) - N(12)^2 = 0$, then

$$\Delta = (6400N^3 + 2560N^4 + 256N^5)(12)^{10} + (1440N + 160N^2 + 3456)(12)^5(15)^5 - (15)^4$$

which is a perfect square of

$$N^5 + 35N^4 + 475N^3 + 3105N^2 + 9720N + 11664 = 0$$

OR

$$(N + 4)^2(N + 9)^3 = 0$$
$$\therefore \quad N = -4 \vee 9$$

and our quintics for these are

$$(z - b_1)^5 - 10(21)(z - b_1)^3 + 20(21)(z - b_1) - (51) = 0$$
$$(z - b_1)^5 - 10(21)(z - b_1)^3 + 45(21)^2(z - b_1) - (51) = 0$$

These give y as

$$(51) = \sqrt[4]{((51)^2 - 128(21)^5)^2}$$
$$(51) = \sqrt[4]{((51)^2 - 1728(21)^5)^2}$$

Now

$$y_1y_2 = 128(21)^5 \ \vee \ 1728(21)^5$$
$$y_3y_4 = 2(51)^2 - 128(21)^5 \ \vee \ 2(51)^2 - 1728(21)^5$$

where both are rational and symmetrically expressible. One is a perfect fifth power. The (21) only appears as $(21)^5$. So they solve quintics where (21) becomes $\omega(21)$, $\omega^2(21)$, $\omega^3(21)$, $\omega^4(21)$ OR where $z - b_1$ is multiplied by these roots of unity. And if y_1,y_2 are qfns of the $(z - b_1)$'s, each is a fifth power, as their product is a fifth power, as that alone allows soln of a set of five quintics. And if they were known to have the irreducible form of

$$(51) \pm \sqrt{((51)^2 - N^5(12)^5)}$$

they would have an extractable root of form $R_1 \pm \sqrt{R_2}$ as the y_1,y_2 must admit the variation allowing soln of the entire set. In both cases $y_1^{1/5}$ and $y_2^{1/5}$ are rational and

$$\omega y_1^{1/5} + \omega^4 y_2^{1/5} \ \text{OR} \ \omega y_1^{1/5} + (N(21))/\omega y_1^{1/5}$$

is a five-valued fn of the roots with values

$$y_1^{1/5} + y_2^{1/5},\ \omega y_1^{1/5} + \omega^4 y_2^{1/5},\ \omega^2 y_1^{1/5} + \omega^3 y_2^{1/5},\ \omega^3 y_1^{1/5} + \omega^2 y_2^{1/5},\ \omega^4 y_1^{1/5} + \omega y_2^{1/5}$$

The eqn in t with these roots is

$$t^5 - 5N(21)t^3 + 5N^2(21)^2 t - 2(51) = 0$$

where t is a fn of the $(z - b_1)$'s with five values. When $N^5 = 128$ the first eqn is

$$t_5 - 10\cdot 2^{2/5}(21)t^3 + 20\cdot 2^{4/5}(21)^2 t - 2(51) = 0$$

Let $t = 2^{1/5}v$ and we have

$$v^5 - 10(21)v^3 + 20(21)^2 v - (51) = 0$$

Here $v = 2^{-1/5}t = (y_1/2)^{1/5} + (y_2/2)^{1/5}$ is equivalent to $(z - b_1)$ and this is its soln. The eqn itself is now De Moivre's form. This y_1 and y_2 are rational and t is a linear fn of the $(z - b_1)$'s with five values by permutations of the roots. For the second eqn, its form in t is

$$t_5 - 10(54)^{1/5}(21)t^3 + 20(54)^{2/5}(21)^2 t - 2(51) = 0$$

We cannot here conclude that this second eqn in z is reducible to the same form as the first where t is a qfn of z with coeffs in terms of (21) and (51). In each of these eqns, our expressions are not unique, but infinitely various. To compare them we need **unique** expressions of these resolvents. We would here have to go back to the given quintic of five parameters of which each eqn in z is a distinct transform.

What we have proved in this digression is that if the complete quintic in χ were reducible to the second form z = ψx, it would be reducible to the first form in t by a further transform in which t was a qfn of z, say χz. Here the coeffs of ψ and χ would be expressible only in terms of coeffs of the general quintic in x and not in terms of (21) and (51). Here our digression ends.

We will approach the quintic by the second method. Consider the dscr of the quintic and cause (12) and (13) to vanish. In the cubic, the dscr was $(13)^2 + N(12)^3$; in the conditioned quartic, $(14)^3 + N(13)^4$; in our proposed conditioned quintic, $(15)^4 + N(14)^5$. N is always $\pm(n - 1)^{n-1}$. With this dscr, we can fulfill a condition of resolubility: any eqn in y where the constant term if a perfect fifth power of a qfn of the coeffs, i.e. $(41)^5$, and will not otherwise contain (41). We need to express (12) = 0 and (13) = 0 in terms of the roots. The proposed conditional quintic is

$$z^5 - 5b_1z^4 + 10b_1^2z^3 - 10b_1^3z^2 + 5b_4z - b_5 = 0$$

and of (21), (31), (41), and (51) only (41) and (51) have significant value. The most complete form of a quartic in y with coeffs as qfns of the quintic's is

$$y^4 - 4N_1(51)y^3 + 6N_2(51)^2y^2 - 4N_3(51)^3y + N_4(41)^5 = 0$$

In the last term, any multiple of (51) is omitted as it could vanish under a proper lin.trans. We need the five N's to make the dscr of the quartic that of the quintic.

$$(I{\cdot}II) = (N_1^2 - N_2)(51)^2 = C(51)^2$$
$$(I{\cdot}III) = (2N_1^3 - 3N_1N_2 + N_3)(51)^3 = D(51)^3$$
$$(I{\cdot}IV) = (3N_1 - 6N_1^2N_2 + 4N_1N_3)(51)^4 - N_4(41)^5 = E(51)^4 - N_4(51)^5$$

So by our dscr table, we have for our dscr:

$$(81C^4E - 54C^3D^2 + 18C^2E^2 - 54CD^2E + 27D^4 + E^3)(51)^{12}$$
$$- (81C^4N_4 + 36C^2N_4E - 54CD^2N_4 + 3E^2N_4)(51)^8(41)^3$$
$$+ (18C^2N_4^2 + 3EN_4^2)(51)^4(41)^{10} - (N_4^3)(41)^{15}$$

which must be compared with

$$((51)^4 - 256(41)^5)^3$$

If we make the arbitrary $N_4 = 256$, we have three eqns:

$$6C^2 + E = 1 \qquad [1]$$
$$27C^4 + 12C^2E - 18CD^2 + E^2 = 1 \qquad [2]$$
$$81C^4E - 54C^3D^2 + 18C^2E^2 - 54CD^2E + 27D^4 + E^3 = 1 \qquad [3]$$

From $[1]^2 - [2]$

$$9C^4 + 18CD^2 = 0$$

One soln is $C = 0 \therefore E = 1, D = 0 \therefore N_1 = N_2 = N_3 = 1$

$$\therefore \qquad (y - (51))^4 + 256(41)^5 - (51)^4 = 0$$

Here there are no redundant eqns or inconsistencies and $C^2 + 2D^2 = 0$ gives the same result. So we can express y in terms of (41),(51) or b_1,b_4,b_5 and therefore with the three z's. But (12) = 0 and (13) = 0 are $2(\sum x)^2 - 5\sum xx = 0$ and $2(\sum x)^3 - 25(\sum xxx) = 0$ and allow infinite forms for y. We need a unique one that represents y as a fn of coeffs or roots of a complete quintic in z by definite relation, i.e. by representing a complete quintic by one containing, as with the quartic, only two parameters.

To achieve a quintic or system of quintics subject to (12) = 0 ∧ (13) = 0, we use the Tschirnhausen-Jerrard Transform (TJ-trans). We take a complete quintic in x

$$x^5 - 5a_1x^4 + 10a_2x^3 - 10a_3x^2 + 5a_4x - a_5 = 0$$

and transform it into a quintic in z where $z = \psi x$. The fullest form for (12) = (13) = 0 is $z = \psi x = hx^4 + kx^3 + lx^2 + mx$ which gives us

$$z^5 - \sum(\psi x)z^4 + \sum(\psi x\psi x)z^3 - \sum(\psi x\psi x\psi x)z^2 + \sum(\psi x\psi x\psi x\psi x)z - \sum\psi x_i \; [1\text{-}5] = 0$$

For (21) = (31) = 0, we take h, k, l m:

$$4(\sum(\psi x))^2 - 10\sum(\psi x\psi x) = 0$$
$$12(\sum(\psi x))^3 - 150\sum(\psi x\psi x\psi x) = 0$$

The first of these eqns conditions is

$$h^2\underline{44} + 2hk\underline{43} + 2hl\underline{42} + 2hm\underline{41} + k^2\underline{33} + 2kl\underline{32} + 2km\underline{31} + l^2\underline{22} + 2lm\underline{21} + m^2\underline{11} = 0$$

By means of the disposable constant, this divides into two linear eqns. The condition now has this form:

$$\begin{aligned}
&(\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})(m\underline{11} + l\underline{12} + k\underline{13} + h\underline{14})^2 \\
&\quad +((\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})l + (\underline{11}\cdot\underline{23} - \underline{12}\cdot\underline{13})k + (\underline{11}\cdot\underline{24} - \underline{12}\cdot\underline{14})h)^2 \\
&\quad +((\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})(\underline{11}\cdot\underline{33} - \underline{13}\cdot\underline{13}) - (\underline{11}\cdot\underline{23} - \underline{12}\cdot\underline{13})^2)k^2 \\
&\quad + 2((\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})(\underline{11}\cdot\underline{34} - \underline{13}\cdot\underline{14}) - (\underline{11}\cdot\underline{23} - \underline{12}\cdot\underline{13})(\underline{11}\cdot\underline{24} - \underline{12}\cdot\underline{14}))hk \\
&\quad +((\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})(\underline{14}\cdot\underline{44} - \underline{14}\cdot\underline{14}) - (\underline{11}\cdot\underline{24} - \underline{12}\cdot\underline{14})^2)h^2 = 0
\end{aligned}$$

This can be made into two parts, using the disposable constants; the first two terms made equal to zero; the last three made **separately** equal to zero. We then have a quad in h,k with rational coeffs (the **first quad**) whose soln gives a linear relation between h and k. The coeff of this relation is a 30° quad-irrational. Substituting for h in the eqn of the sum of the two first terms made equal to zero gives another quad eqn with quad -irrational coeffs (the **second quad**). Its soln gives a linear relation between m,l,k whose coeff is a quad-quad-irrational. Substituting these in the cubic eqn of condition which is

$$\begin{aligned}
&h^3\underline{444} + 3h^2k\underline{443} + 3h^2l\underline{442} + 3h2m\underline{441} \\
&\quad + 3hk^2\underline{433} + 6hkl\underline{432} + 6hkm\underline{431} \\
&\quad + 3hl^2\underline{422} + 6hlm\underline{421} \\
&\quad + 3hm^2\underline{411} \\
&\quad + k^2\underline{333} \quad + 3k^2l\underline{332} + 3k^2m\underline{331} \\
&\quad + 3kl^2\underline{322} + 6klm\underline{321} \\
&\quad + 3km^2\underline{311} \\
&\quad + l^3\underline{222} \quad + 3l^2m\underline{221} \\
&\quad + 3lm^2\underline{211} \\
&\quad + m^3\underline{111} \quad = 0
\end{aligned}$$

The result takes this form:

$$A_1m^3 + 3A_2m^2l + 3A_3ml^2 + A_4l^3 = 0$$

where the A_i are quad-quad-irrational fns of the a_i [1-5]. Soln of this cubic gives the relation of m and l as an irrational fn of the coeffs of the original quintic. This fn comes from the resolution of a cubic with coeffs derived from the soln of a given quad with coeffs of irrational fns from another quad with coeffs of qfns of a_i [1-5]. This would make it a cubic-quad-quad-irrational. With this, ψx is determined giving us b_1,b_4,b_5 by subbing ψ into the sym. fns of the coeffs of the transformed eqn.

Now h in terms of k comes from $A_{16}h = k(A_{15} + \sqrt{I_{30}})$, the suffixes being the degree of the known qfn of the latter. Subbing this for h in the cubic:

$$\begin{aligned}
&m^3\underline{111}A^3_{16} + 3m^2l\underline{112}A^3_{16} + 3m^2k(\underline{113}A_{16} + \underline{114}A_{15} + \underline{114}\sqrt{I_{30}})A^2_{16} \\
&\quad + 3ml^2\underline{122}A^3_{16} + 6mlk(\underline{123}A_{16} + \underline{124}A_{15} + \underline{124}\sqrt{I_{30}})A^2_{16} \\
&\quad + 3mk^2(\underline{133}A^3_{16} + \underline{134}A^2_{16}A_{15} + \underline{144}A_{16}(A^2_{15} + I_{50}) + (\underline{134}A^2_{16} \\
&\quad + 2\underline{144}A_{16}A_{15})\sqrt{I_{30}}) + l^2\underline{222}A^3_{16} + 3l^2k(\underline{223}A^3_{16} + \underline{224}A^2_{16}A_{15} \\
&\quad + \underline{224}A^2_{16}\sqrt{I_{30}}) + 3lk^2(\underline{233}A^3_{16} + \underline{234}A_{16}A_{15} + \underline{244}A_{16}(A^2_{15} + I_{30}) \\
&\quad + (\underline{234}A^2_{16} + 2\underline{224}A_{16}A_{15})\sqrt{I_{30}}) + k^3(\underline{333}A^3_{16} + 3\underline{334}A^2_{16}A_{15} \\
&\quad + 3\underline{344}A_{16}(A^2_{15} + I_{30}) + \underline{444}(A^3_{15} + 3A_{15}I_{30}) + (3\underline{334}A^2_{16} \\
&\quad + 6\underline{344}A_{16}A_{15} + \underline{444}(3A^2_{15} + I_{30})\sqrt{I_{30}})
\end{aligned}$$

Then k in terms of l comes from

$$\underline{11}A_{16}M + (\underline{12} + \sqrt{A_6})A_{16}l + (A_{16}\underline{13} + A_{15}\underline{14} + \sqrt{I_{30}}\underline{14})k = 0$$

where $A_6 = -(\underline{11}\cdot\underline{22} - \underline{12}\cdot\underline{12})$ and, making the denoms rational:

$$k((\underline{13}A_{16} + \underline{14}A_{15})^2 - \underline{14}^2I_{30}) + A_{16}(\underline{13}A_{16} + \underline{14}A_{15} + \underline{14}\sqrt{I_{30}})(\underline{11}m + (\underline{12} + \sqrt{A_6})l) = 0$$

which is also

$$A_{40}k + A_{16}(\underline{13}A_{16} + \underline{14}A_{15} + \underline{14}\sqrt{I_{30}})(\underline{11}m + (\underline{12} + \sqrt{A_6})l) = 0$$

Then the complete expression of the cubic is

$$
\begin{aligned}
&m^3(\underline{111}A^3_{40} + 3\underline{11}A^2_{40}B_{41} + 3\underline{11}^2A_{40}B_{79} + \underline{11}^3B_{117} \\
&\quad + (3\underline{11}A^2_{40}B_{26} + 3\underline{11}^2A_{40}B_{64} + B_{102})\sqrt{I_{30}}) \\
&\quad + 3m^2l(\underline{112}A^3_{40} + (\underline{12} + \sqrt{A_6})A^2_{40}B_{41} + 2\underline{11}A^2_{40}B_{42} \\
&\quad + 2\underline{11}(\underline{12} + \sqrt{A_6})A_{40}B_{79} + \underline{11}^2A_{40}B_{80} + \underline{11}^2(\underline{12} + \sqrt{A_6})B_{117} \\
&\quad + (A^2_{40}(\underline{12} + \sqrt{A_6})B_{26} + 2\underline{11}A^2_{40}B_{27} + 2\underline{11}(\underline{12} + \sqrt{A_6})A_{40}B_{64} \\
&\quad + \underline{11}^2A_{40}B_{65} + \underline{11}^2(\underline{12} + \sqrt{A_6})B_{102})\sqrt{I_{30}}) \\
&\quad + 3ml^2(\underline{122}A^3_{40} + 2(\underline{12} + \sqrt{A_6})A^2_{40}B_{42} + \underline{11}A^2_{40}B_{43} \\
&\quad + (\underline{12} + \sqrt{A_6})^2A_{40}B_{79} + 2\underline{11}(\underline{12} + \sqrt{A_6})A_{40}B_{80} + \underline{11}(\underline{12} + \sqrt{A_6})^2B_{117} \\
&\quad + (2(\underline{12} + \sqrt{A_6})A^2_{40}B_{79} + \underline{11}A^2_{40}B_{28} + (\underline{12} + \sqrt{A_6})^2A_{40}B_{64} \\
&\quad + 2\underline{11}(\underline{12} + \sqrt{A_6})A_{40}B_{45} + \underline{11}(\underline{12} + \sqrt{A_6})^2B_{10})\sqrt{I_{30}}) \\
&\quad + 3l^2(\underline{222}A^3_{40} + 3(\underline{12} + \sqrt{A_6})A^2_{40}B_{43} + 3(\underline{12} + \sqrt{A_6})^2A_{40}B_{80} \\
&\quad + (\underline{12} + \sqrt{A_6})^3B_{117} + (3(\underline{12} + \sqrt{A_6})A^2_{40}B_{28} + 3(\underline{12} + \sqrt{A_6})^2A_{40}B_{64} \\
&\quad + (\underline{12} + \sqrt{A_6})^3B_{102})\sqrt{I_{30}})
\end{aligned}
$$

where the B's *are rational and not difficult to express*. If we multiply this cubic by the coeff of m^3, using $-\sqrt{I_{30}}$ for $\sqrt{I_{30}}$, and make l equal to that rational product, we can determine m in a whole form by solving the cubic. The coeffs of this cubic, with their quad-quad-irrationality, subbed into the roots of a general cubic eqn gives us the kind of irrational we consider from this point. We call it irreducible as we need not reduce it and safely conjecture it to be in fact irreducible when expressed in terms of the x's. Given m, *there is no further difficulty in determining h and k*. The result is that a general quintic is equivalent to a system of twelve conditioned quintics of form

$$z^5 - 5b_1z^4 + 10b_1^2z^3 - 10b_1^3z^2 + 5b_4z - b_5 = 0$$

with the relation $z = hx^4 + kx^3 + lx^2 + mx$ where k:h, m:h, b_1, b_4, b_5 are cubic-quad-quad-irrational fns of a_i [1-5], all with the same kind of irrationality and all 12-valued from the square- and cube-roots of unity we use to solve the eqns giving rise to the irreducible fn.

The above method is the simplest approach. The two quad-irrationals are of degrees 6° and 30°; the highest degree in the cubic, 126°, which becomes 249° when an extreme coeff is rationalized. Eliminating l and m for a cubic in h and k, the degrees become 14, 50, 192, 384 respectively. There are six ways to vary the elimination, the other four giving intermediate results of degree. Each method gives twelve conditioned quintics and a relation between x and z.

Consider one of the conditioned quintics and its quartic in y with coeffs of qfns of the quintic's and sharing its dscr. We have, by the quartic's resolution, a complete expression in y in terms of a_i [1-5] involving our special irrationality. Subbing the sym. fns of x into the coeffs, we express y in terms of x and for each mode of transformation this is unique.

Our quintic in z (ψx) and the quartic in y have the same critical fns which here is only the dscr as the quintic has only two parameters and therefore only one condition. By this, if the quartic in y has two equal roots, it has four equal roots. The dscr of the quintic in x is a factor of the one for the quintic in z and the latter vanishes when the former does. When the quintic in z has more than two equal roots, it takes this form: $(\psi x - b_1)^5 = 0$ and the quartic is $y^4 = 0$. If we suppose such conditions to exist between the coeffs of the original quintic as would be inconsistent with the form of the quartic in z, the transformation would need to be modified or might even be nullified.

Regarding the rationality of the y's in the quartic, they differ from our earlier cases. We required that the roots of a quantic be independent of each other. But in our quartic in y, any two are linear expressions of the other two. If y_1, y_2 are taken as independent, then y_3, y_4 each take form $N_1y_1 + N_2y_2$. So our argument as to the rationality fo y fails us. As we have not four but two independent elements in the resolvent, we conclude that if the roots are to be rational, it results in the form of a quadratic.

Let $\Delta = (51)^4 - 256(41)^5$ and we take the quartic's roots as:

$$y_1 = (51) + \sqrt[4]{\Delta}$$
$$y_2 = (51) - \sqrt[4]{\Delta}$$
$$y_3 = (51) + i\sqrt[4]{\Delta}$$
$$y_4 = (51) - i\sqrt[4]{\Delta}$$

which are rational (but not sym.) fns of z and of $y_1 \cdot y_2$, $y_3 \cdot y_4$ their product is $256(41)^5$, a fifth power rational and sym. fn of the z's. If we call these products Y_1 and Y_2, they are the roots of

$$Y^2 - 2(51)^2Y + 256(41)^5 = 0$$

Y_1,Y_2 are qfns of the z's and $Y_1 \cdot Y_2$ is a perfect fifth power of a rational fn -- (41) -- of the z's. So Y is a perfect fifth power qfn of 2° of the z's as any other form is excluded. Calculation of each Y gives two 10° expressions taking the form

$$\sum(A_1t^2) + \sum(B_1tt) = Y_1^{1/5}$$
$$\sum(A_2t^2) + \sum(B_2tt) = Y_2^{1/5}$$

where $t = z - b_1 = \psi x - b_1$. These Y_i are known fns of a_i [1-5] and A_i,B_i arise in the calculation. This gives us two unsymmetrical fns of 2° of t and shows the quartic resoluble. These, resting on the perfect square of the dscr, must be rational fns of the x's. This gives us these fns for one form of ψ and we may obtain Sym. the fns of the other forms of ψ: i.e. ψ_{212} comes from the second root of the cubic, first root of the second quad, second root of the first quad. If we form the fns Y for suffixes 111, 211, 311 and combine them in any sym. manner, we get two fns corresponding to Y where the cubic's irrationality is gone and these new fns, χ, contain only the irrationality from the two quads. Let these by χ_{11}, Sym. derive fns χ_{21} and combining symmetrically, we get fn λ_1 with only the irrationality of the first quad. Sym. derive λ_2, combine the λ's symmetrically and we have fns μ which are qfns of the x's.

If all our sym. combinations are simple sums, our two non-symmetric qfns of the roots are

$$\sum(A_1(\mu x)^2 + \sum(B_1\mu x\mu x) = S(Y_1^{1/5})$$
$$\sum(A_2(\mu x)^2 + \sum(B_2\mu x\mu x) = S(Y_2^{1/5})$$

We can lose any powers of x above four and eliminate some of the x's by means of the above eqns. Then $\sum t = 0$, $\sum(tt) = 0$, $\sum(ttt) = 0$. We can then derive the quintic's roots without further radicals. These roots are completely determined and the only extractions are the $Y^{1/5}$'s. All twenty-four of these are of the same form with different combinations of square- and cube-roots of unity. Note that we can determine $\sqrt{\Delta}$ without forming Δ as $\sqrt{\Delta}$ is the square root of the dscr multiplied by ten expressions, one being

$$h(x_1^3 + x_1^2x_2 + x_1x_2^2x_3^3) + k(x_1^2 + x_1x_2 + x_2^2) + l(x_1 + x_2) + m$$

Note also that we did not solve the general quintic from its trinomial form. We see that the principle of soln is **not** that the roots of the resolvent are expressible as qfns of the roots of the given quantic. *The principle is, that of whatever order of irrationality (including rationality) the root of the resolvent is, when expressed in terms of the roots of the quantic, its prime-root will also be of the same order of irrationality, or rational, as the case may be.*

If n is the degree of the quantic (n assumed prime), then value of y is of the form that its nth root is expressible as n values of $1^{1/n}P$, where P is of the same radical form as y but each member to the 1/nth degree of that in y. In the cubic, y is rational and 3°, therefore $y^{1/3}$ is rational and linear. In the quintic, y takes form $S_5 + \sqrt{R_{10}}$. Therefore $y^{1/5}$ has form $R_1 + \sqrt{R_2}$ where S is symmetric and the R's are qfns of ψx's. The quartic resolvent is divisible into two quads, in each of which the fifth-root of the product of its roots is rational as indicated above. And the product of two corresponding roots in these quads is rational and symmetric in the same sense. Or we can say there is a connection between the four values of $y^{1/5}$ where they divide into two pairs, the product of whose corresponding member is a qfn of the five values of ψx or rather, a qfn of the five x's equivalent to the same qfn of the ψx's.

We denote by a **De Moivrean Equation** a quantic of prime p° with roots of form $\omega P + \omega^{p-1}Q$, PQ being a symmetric, one-valued fn of the roots. Our resolution by one quad resolvent is reduction to De Moivrean form. In the cubic roots are $\omega P + \omega^2 Q$ where PQ is sym. in the roots and rational in the coeffs. In the first two quintics in our digression, each root, or a qfn of each root, has form $\omega P + \omega^4 Q$ where PQ is a qfn of the coeffs. The peculiarity here is that $\omega P + \omega^{p-1}Q$ is $\omega P + S/\omega P$, an expression of p values and must therefore be a qfn of some one root.

In the general quintic, expressed in ψx, PQ is not sym. in the roots and not rational in the coeffs. It is demi-symmetrical in the roots and involves a square-root radical which expressed in the coeffs.

If we make the five values of $\omega y_1^{1/5} + \omega^4 y_2^{1/5}$ the roots of the De Moivrean quintic

$$t^5 - 5At^3 + 5A^2t - 2(51) = 0$$

A is $(y_1y_2)^{1/5}$, a two-valued qfn of the ψx's in form P + √Q in the coeffs and doing the same with y_3,y_4 in

$$t^5 - 5Bt^3 + 5B^2t - 2(51) = 0$$

B is $(y_3y_4)^{1/5}$ with Sym. results. And each A of form P + √Q has a B as P - √Q with the same P and Q in both. Then $\omega y_1^{1/5} + \omega^4 y_2^{1/5}$ and $\omega y_3^{1/5} + \omega^4 y_4^{1/5}$ are five-valued fns, through a quad-radical and are of opposite sign. And the five values of our

$$\omega Y_1^{1/5} + \omega^4 Y_2^{1/5}$$

are the roots of the De Moivrean quintic with coeffs as sym. fns of the z's:

$$t^5 - 5\cdot 256^{1/5}(41)t^3 + 5\cdot(256^{1/5})^2(41)^2t - 2(51)^2 = 0$$

This system of Y might not completely determine the roots. But if we consider every possible value of $\omega^p y_1^{1/5} + \omega^q y_2^{1/5}$, there are five and they are linear fns of the z's. These are $y_1^{1/5} + y_2^{1/5}$ and this multiplied by each fifth-root of unity. Sym. for y_3,y_4. We arrive at five systems of two eqns, using t = ψx - b_1:

$$(y_1^{1/5} + y_2^{1/5}) = \sum(At)$$
$$(y_3^{1/5} + y_4^{1/5}) = \sum(Bt)$$
$$\omega(y_1^{1/5} + y_2^{1/5}) = \sum(At)$$
$$\omega^4(y_3^{1/5} + y_4^{1/5}) = \sum(Bt)$$
$$\omega^2(y_1^{1/5} + y_2^{1/5}) = \sum(At)$$
$$\omega^3(y_3^{1/5} + y_4^{1/5}) = \sum(Bt)$$
$$\omega^3(y_1^{1/5} + y_2^{1/5}) = \sum(At)$$
$$\omega^2(y_3^{1/5} + y_4^{1/5}) = \sum(Bt)$$
$$\omega^4(y_1^{1/5} + y_2^{1/5}) = \sum(At)$$
$$\omega(y_3^{1/5} + y_4^{1/5}) = \sum(Bt)$$

Repeat this system for each of the twelve forms of t ≡ψx - b_1 with the corresponding forms of y an we have 120 eqns, combinable symmetrically in twelve to produce these systems:

$$S(y_1^{1/5}) + S(y_2^{1/5}) = \sum(A\lambda x)$$
$$S(y_3^{1/5}) + S(y_4^{1/5}) = \sum(B\lambda x)$$

and the four derived by multiplying these by each fifth-root of unity. Here λx is a 4° qfn of a_i [1-5] as coeffs. We cannot algebraically determine which system goes with a given quintic as these solve a set of five conjugate quintics. But here, Y and y determine the roots.

Remarks

The number of values a complete fn of the roots of an n° quantic has is n! and this is the number of elements in its soln. In a cubic, we have 2·3 eqns, each with one radical. In our first method for the quartic, we had 2·3 eqns each with two radicals for 12 elements. In the second complete method, 2·2·3 eqns, each with two radicals and 24 = 4! elements. In the quintic, 2·5 eqns containing 2·2·3 radicals for 120 elements.

In each case we use a resolvent whose roots enter into the algebraic expression of the roots of the quantic. In the cubic, the resolvent is a quadratic with rational coeffs. In the quartic, the first method uses a 3° qfn, the second, a 6° qfn divisible into two cubics with quad-irrational coeffs. In the quintic, it is a 24° qfn divisible into 12 quads with our special irrational in the coeffs. In every case, the final term of the resolvent is of the same power as the quantic itself, a qfn of that quantic's coeffs. From this we can infer the solubility of the conjugates.

Abel's Theorem *If a root is expressible as an irreducible irrational fn of the coeffs, every radical which enters into the composition of the fn is expressible as a qfn of the roots and this qfn has the same multiplicity of value by transposition of the roots, as the irrational has by reason of the different roots of unity which are implied in its radicals.*

Although this remains true of fns soluable by the first method using linear transformation, it is a subset of the truth when the second method is considered. Entering this more deeply, Hargreave considers this theorem, at bottom, to be a mere truism, a circuitous definition of resolution. Hamilton's theorem that all methods of quartic soln are substantially the same also remains true for those where the first method is applicable. Hargreave expands on the relation of his proven theory with respect to these and other theorems. I leave this to your curiosity. It goes (almost) without saying that Hargreave's theory must stand in a particular relation to Galois Theory. Under Hargreave's theory, as I grasp it, if a quintic is algebraically resoluble it has a quadratic resolvent where if the general quintic can be transformed into a quintic with two parameters -- (41), (51) -- and whose discriminant, roots, and coeffs are susceptible of 12 values, then its resolution becomes possible. I leave Hargreave's relation to Galois to more capable hands. I have studied three texts of Galois Theory and my response so far is that of the French mathematician who said, in effect, of Group Theory, "I see what you are doing. But I don't see why."

In an appendix, Hargreave notes that a quintic can also be solved through a cubic resolvent. The dscr of the quintic here can be represented as the sum of a fifth and a third power. If (12) = (14) = 0 we have as the dscr

$$(15)((15)^3 - 3456(13)^5)$$

and because (15) only vanishes in extreme cases, the condition of our having two equal roots can be

$$(15)^3 - 3456(13)^5 = 0$$

With the TJ-trans, the general quintic has a form with only (13) and (15). As the eqn is conditioned, we write for this (31) and (51) setting (12) = (14) = 0. We divide the condition of (21) into two parts as we did earlier, and substituting the values in (41) = 0, we have

$$\underline{4444}h^4 + \underline{4443}h^3k + \cdots$$

Solving this quartic gives us h, k, l, and m. We find from this that we now need 16 forms of ψx instead of 12. This gives 16 quintics of form:

$$(\psi x - b_1)^5 - 10(31)(\psi x - b_1)^2 - (51) = 0$$

and our cubic resolvents for these have form

$$y^3 - 3N_1(51)y^2 + 3N_2(51)^2y - N_3(31) = 0$$

By the same method as earlier but equating the dscr of the cubic to the square of the quintic's, we have

$$N_1 = N_2 = 1 \quad N_3 = 3456 \ \therefore \ (y - (51))^3 + (51)^3 - 34546(31)^5 = 0$$

OR

$$y = (51) + \sqrt[3]{(3456(31)^5 - (51)^3)}$$

which has all the qualifications of a resolvent. Its last term is a fifth power of a sym. qfn of the $(\psi x - b_1)$'s which fn only enters as a fifth power. So the eqn in y applies to the sent of quintics determined with (31), ω(31), $\omega^2(31)$, $\omega^3(31)$, and $\omega^4(31)$ as coeffs of the middle term. So $y^{1/5}$ is capable of extraction in the form R_1 if y is rational. Else it is of form $R_1 + \sqrt[3]{R_3}$ if the cubic in y is irreducible.

Hargreave summarizes his ideas as follows.

1) The method of proceeding by steps, based on the idea that the algebraic expression of the root of a complete eqn must contain the roots of an eqn of lower degree, all being qfns of the roots of the complete eqn is the foundation of this theory. The soln of a complete eqn involving the soln of all eqns of lower degree implies the necessity of a resolvent whose critical fns are those of the complete eqn.

2) This resolvent applies to a set of quantics including the given complete eqn. And this set has no common roots. The root of the resolvent expressed in terms of the given quantic is a perfect power of another expression similar in rationality, the exponent of the power being the number of elements in the set it must apply to. If this power is p, the pth root of the root of the resolvent must be the same form as the root itself, subbing for each sym. fn of the roots of the quantic of degree mp, a qfn of the same roots of m°.

3) The quantic is transformed in such a way that the form of the root is altered but the root still applies to the original quantic in its complete, unaltered form. This property produces the **uniqueness** in the expression of the root of the resolvent when expressed in terms of the roots of the quantic itself. We cannot use the resolvent unless we can express its roots in a perfectly unique manner and this is only possible in terms of roots or coeffs of an eqn between whose roots or coeffs no special relation exists. For this we must go back to the complete quantic in x.

4) Our transformations diminish the parameters while maintaining a general soln. Given $\varphi(u,v,w,\ldots) = 0$, we find $u = \psi(v,w,\ldots)$ so that of $u = \varphi v$ we can find φ^{-1} in v equivalent to $\varphi^{-1}u$ and not qualify its scope. This is applicable to eqns up to fifth degree. *Beyond this point we cannot go in this direction; for we have arrived at the place where there is necessarily a fundamental change is the very statements and conditions of the problem. This I conceive is the real barrier to algebraic resolution: the impossibility of inventing any problem of two parameters, or any problem of simple inversion, which shall be an adequate representative of a complete algebraic equation of the sixth of higher degrees.* I take this to mean, given what follows, that as of his writing no adequate transform comparable to the T-trans and TJ-trans was available.

Hargreave emphasizes that his method must apply to the complete and general eqn of each degree. In the case of the quintic, the De Moivrean form allows us to do this. We do not conclude that $x^3 - 5a_4x - a_5 = 0$ is reducible to the De Moivrean Equation. What we have proven is that **if** in a trinomial form of $(z - b_1)^5 + 5(41)(z - b_1) - (51) = 0$, its b_1, (41), (51) are fns of the five parameters of the general quintic in x, then the quintic in z is a transformation of, and an equivalence to, the general quintic in x. Then we can find its De Moivrean form with roots adequate to our purpose. The distinction is that a simple transformation to trinomial form has no inverse. In the quartic, a transformation to $z^4 - 4(31)z + (41) = 0$ has two parameters and we can express (31) and (41) in terms of a_i [1-4]. But we cannot do this with $z^4 - 4b_3z + b_4 = 0$ and reach a general quartic with four parameters. In this method, everything we do must retain the necessary relations of the given complete equation's roots, the roots of the transformed equation, and the roots of the resolvent. In this is bound up our requirement of uniqueness. The general quintic **being reducible to Jerrard's form** is also reducible to the De Moivrean. But Jerrard's form alone is **not** reducible to the De Moivrean.

Let us examine the general case of the trinomial, where for n prime

$$(z - b_1)^n + nb_{n-1}(z - b_1) - b_n = 0$$

Its dscr is

$$b_n^{n-1}(n-1)^{n-1}b_{n-1}^{n}$$

and we can use this to write down the De Moivrean quantic with the same discriminant. If we take the quad in Y with roots

$$\left(b_n^{(n-1)/2} \pm \sqrt{(b^{n-1} - (n-1)^{n-1}b_{n-1}^{n})} \right)^{1/n}$$

and then form the De Moivrean eqn whose roots are the n values of $\omega Y_1 + \omega^{n-1}Y_2$, we then have a De Moivrean in t. And we might suppose it to be reducible. But it is not. If we tried to form the values of Y, which should be qfns of the quantic in $z - b_1$ of degree (n-1)/2, we must first find Y^n expressible in terms of b_{n-1} and b_n. But what of the corresponding expressions of b_{n-1} and b_n in terms of the roots? The n-2 conditions among the roots turn every expression of them into an n-fold infinity of expression and preclude uniqueness. We cannot then uniquely relate Y or t to the z's.

But if the trinomial quantic in $z - b_1$ is a proper transform of and therefore equivalent to our n° complete quantic in x with n independent parameters a_i [1-n], then we can express Y_1 and Y_2 in terms of the coeffs in a completely unique manner. The square root radical in this Y is *a priori* reducible and the nth root would be extractable in fns of ((n-1)/2)° of the fn of x which we denote by z. We could then form the De Moivrean in t. And if this did not absolutely lead to resolution, it would determine non-symmetric fns of the roots presumably leading to resolution.

In point of theory, the general quantic **is** reducible to trinomial form by Tschirnhausen's process although the practical limits of this process are rather limited. But we are not entitled to conclude this makes such quantics irresoluable.

Now what if our quantic in trinomial form is not complete but conditioned? We must still bring it from any form of

$$(z - b_1)^5 + 5b_4(z - b_1) - b_5 = 0$$

to the proper form of

$$(z - b_1)^5 + 5(41)(z - b_1) - (51) = 0$$

where (41),(51) are fns of the five parameters of the general quintic in x. The one form is rational and single-valued. The other has coeffs with cubic-quad-quad irrationals and is twelve-valued. The only form soluble without transformation is the De Moivrean which is in itself a universal resolvent.

Algebraically considered, the general quantic **is** reducible to trinomial form. Suppose it reduced to

$$(z - b_1)^n + (n-1)(n-1\ 1)(z - b_1) - (n\ 1) = 0$$

where z is

$$hx^{n-1} + kx^{n-2} + lx^{n-3} + \cdots + \lambda x$$

and h,k,l,…,λ and (n-1 1),(n 1) are uniquely and exclusively expressed in terms of x_i [1-n] which are the roots. Theoretically, these have (n-2)! values, practically considered, twice that many. The resolvent in y is

$$(y - (n\ 1))^{n-1} - (n\ 1)^{n-1} + (n-1)^{n-1}(n-1\ 1)^n = 0$$

and its roots divisible into (n-1)/2 pairs of form

$$(n\ 1) \pm 1^{(n-1)/2\ n-1}\sqrt{\Delta}$$

whose nth roots have form

$$R_1 \pm 1^{(n-1)/2\ n-1}\sqrt{R_{n-1}}$$

multiplied by all the nth-roots of unity. Let

$$y_1, y_2 = R_1' \pm \omega^{\ n-1}\sqrt{R_{n-1}'}$$
$$y_3, y_4 = R_1'' \pm \omega^{2\ n-1}\sqrt{R_{n-1}''}$$
$$\ldots$$
$$y_{n-2}, y_{n-1} = R_1^{(n-1)/2} + \omega^{(n-1)/2\ n-1}\sqrt{R_{n-1}^{(n-1)/2}}$$

and for each of these we take all values of

$$\omega y_{2p-1} + \omega^{n-1} y_{2p}$$

for the n values of ω, we have n^2 values of which exactly n are linear qfns of the z's. These are y_{2p-1}+ y_{2p} multiplied by the roots of unity. We then have n systems of eqns each with (n-1)/2 elements

$$y_1^{1/n} + y_2^{1/n} = \sum(At)$$
$$y_3^{1/n} + y_4^{1/n} = \sum(Bt)$$
$$\ldots$$
$$y_{n-2}^{1/n} + y_{n-1}^{1/n} = \sum(Lt)$$

and each multiplied by the nth-roots of unity. We repeat this for all forms of t which is $\psi x - b_1$ and this gives us (1,2,…,n) eqns. We combine them as before, solving our quantic and the n-1 other conjugate quantics.

This requirement that a quantic, to be resoluble, need be reduced to a particular trinomial form is connected to the work of Harley and Boole as to the binomial form of differential eqns resulting from trinomials. For a trinomial quantic, the problem is to find its inversion, finding t in terms of v from v = φt. In quantics of composite degree, the eqn in y still holds when one term is given the opposite sign when n is even. The trinomial form has really only one parameter. Multiplying this parameter by the roots of unity gives the same result. No other multinomial of more parameters has this property. A De Moivrean Equation of prime degree n has this property. And this suggests that the trinomial form and the De Moivrean form have some conjugate relation between them, as suggested in part by Jerrard.

I should add that all of this last chapter, apart from a few remarks, is solely Hargreave. All the credit of these ideas goes to him.