CHAPTER X.

PROBABILITY AND THEORY OF ERRORS.

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ART. 1. INTRODUCTION.

It is a curious circumstance that a science so profoundly mathematical as the theory of probability should have originated in the games of chance which occupy the thoughtless and profligate.* That such is the case is sufficiently attested by the fact that much of the terminology of the science and many of its familiar illustrations are drawn directly from the vocabulary and the paraphernalia of the gambler and the trickster. It is somewhat surprising, also, considering the antiquity of games of chance, that formal reasoning on the simpler questions in probability did not begin before the time of Pascal and Fermat. Pascal was led to consider the subject during the year 1654 through a problem proposed to him by the Chevalier de Méré, a reputed gambler.† The problem in question is known as the problem of points and may be stated as follows: two players need each a given number of points to win at a certain stage of their game; if they stop at this stage, how should the stakes be divided? Pascal corresponded with his friend Fermat on this question; and it appears that the letters which passed between them contained the earliest distinct formulation of principles falling within the theory of probability. These

* The historical facts referred to in this article are drawn mostly from Todhunter's History of the Mathematical Theory of Probability from the time of Pascal to that of Laplace (Cambridge and London, 1865).

† "Un problème relatif aux jeux de hasard, proposé à un austère janséniste par un homme du monde, a été l'origine du calcul des probabilités." Poisson, Recherches sur la Probabilité des Jugements (Paris, 1837).

acute thinkers, however, accomplished little more than a correct start in the science. Each seemed to rest content at the time with the approbation of the other. Pascal soon renounced such mundane studies altogether; Fermat had only the scant leisure of a life busy with affairs to devote to mathematics; and both died soon after the epoch in question,—Pascal in 1662, and Fermat in 1665.

A subject which had attracted the attention of such distinguished mathematicians could not fail to excite the interest of their contemporaries and successors. Amongst the former Huygens is the most noted. He has the honor of publishing the first treatise* on the subject. It contains only fourteen propositions and is devoted entirely to games of chance, but it gave the best account of the theory down to the beginning of the eighteenth century, when it was superseded by the more elaborate works of James Bernoulli,† Montmort,‡ and De Moivre.§ Through the labors of the latter authors the mathematical theory of probability was greatly extended. They attacked, quite successfully in the main, the most difficult problems; and great credit is due them for the energy and ability displayed in developing a science which seemed at the time to have no higher aim than intellectual diversion.‖ Their names, undoubtedly, with one exception, that of Laplace, are the most important in the history of probability.

Since the beginning of the eighteenth century almost every mathematician of note has been a contributor to or an expositor of the theory of probability. Nicolas, Daniel, and John Bernoulli, Simpson, Euler, d'Alembert, Bayes, Lagrange, Lambert, Condorcet, and Laplace are the principal names which figure in the history of the subject during the hundred years

* De Ratiociniis in Ludo Aleæ, 1657.

† Ars Conjectandi, 1713.

‡ Essai d'Analyse sur les Jeux de Hazards, 1708.

§ The Doctrine of Chances, 1718.

‖ Todhunter says of Montmort, for example, "In 1708 he published his work on Chances, where with the courage of Columbus he revealed a new world to Mathematicians."

ending with the first quarter of the present century. Of the contributions from this brilliant array of mathematical talent, the Théorie Analytique des Probabilités of Laplace is by far the most profound and comprehensive. It is, like his Mécanique Céleste in dynamical astronomy, still the most elaborate treatise on the subject. An idea of the grand scale of the work in its present form* may be gained by the facts that the non-mathematical introduction† covers about one hundred and fifty quarto pages; and that, in spite of the extraordinary brevity of mathematical language, the pure theory and its accessories and applications require about six hundred and fifty pages.

From the epoch of Laplace down to the present time the extensions of the science have been most noteworthy in the fields of practical applications, as in the adjustment of observations, and in problems of insurance, statistics, etc. Amongst the most important of the pioneers in these fields should be mentioned Poisson, Gauss, Bessel, and De Morgan. Numerous authors, also, have done much to simplify one or another branch of the subject and thus bring it within the range of elementary presentation. The fundamental principles of the theory are, indeed, now accessible in the best text-books on algebra: and there are many excellent treatises on the pure theory and its various applications.

Of all the applications of the doctrine of probability none is of greater utility than the theory of errors. In astronomy, geodesy, physics, and chemistry, as in every science which attains precision in measuring, weighing, and computing, a knowledge of the theory of errors is indispensable. By the aid of this theory the exact sciences have made great progress dur-

*The form of the third edition published in 1820, and of Vol. VII of the complete works of Laplace recently republished under the auspices of the Académie des Sciences by Gauthier-Villars. This Vol. VII bears the date 1886.

† "Cette Introduction," writes Laplace, "est le developpement d'une Leçon sur les Probabilités, que je donnai en 1795, aux Écoles Normales, où je fus appelé comme professeur de Mathématiques avec Lagrange, par un décret de la Convention nationale."

ing the present century, not only in the actual determination of the constants of nature, but also in the fixation of clear ideas as to the possibilities of future conquests in the same direction. Nothing, for example, is more satisfactory and instructive in the history of science than the success with which the unique method of least squares has been applied to the problems presented by the earth and the other members of the solar system. So great, in fact, are the practical value and theoretical importance of the method of least squares, that it is frequently mistaken for the whole theory of errors, and is sometimes regarded as embodying the major part of the doctrine of probability itself.

As may be inferred from this brief sketch, the theory of probability and its more important applications now constitute an extensive body of mathematical principles and precepts. Obviously, therefore, it will be impossible within the limits of a single chapter of this volume to do more than give an outline of the salient features of the subject. It is hoped, however, in accordance with the general plan of the volume, that such outline will prove suggestive and helpful to those who may come to the science for the first time, and also to those who, while somewhat familiar with the difficulties to be overcome, have not acquired a working knowledge of the subject. Effort has been made especially to clear up the difficulties of the theory of errors by presenting a somewhat broader view of the elements of the subject than is found in the standard treatises, which confine attention almost exclusively to the method of least squares. This chapter stops short of that method, and seeks to supply those phases of the theory which are either notably lacking or notably erroneous in works hitherto published. It is believed, also, that the elements here presented are essential to an adequate understanding of the well-worked domain of least squares.*

* The author has given a brief but comprehensive statement of the method of least squares in the volume of Geographical Tables published by the Smithsonian Institution, 1894.

ART. 2. PERMUTATIONS.

The formulas and results of the theory of permutations and combinations are often needed for the statement and solution of problems in probabilities. This theory is now to be found in most works on algebra, and it will therefore suffice here to state the principal formulas and illustrate their meaning by a few numerical examples.

The number of permutations of n things taken r in a group is expressed by the formula

$$(n)_r = n(n-1)(n-2)\dots(n-r+1). \qquad (1)$$

Thus, to illustrate, the number of ways the four letters a, b, c, d can be arranged in groups of two is $4 \cdot 3 = 12$, and the groups are

ab, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, *dc*.

Similarly, the formula gives for

$n = 3$ and $r = 2$, $(3)_2 = 3 \cdot 2 = 6$,
$n = 7$ " $r = 3$, $(7)_3 = 7 \cdot 6 \cdot 5 = 210$,
$n = 10$ " $r = 6$, $(10)_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200$.

The results which follow from equation (1) when n and r do not exceed 10 each are embodied in the following table:

VALUES OF PERMUTATIONS.

	10	9	8	7	6	5	4	3	2	1
1	10	9	8	7	6	5	4	3	2	1
2	90	72	56	42	30	20	12	6	2	
3	720	504	336	210	120	60	24	6		
4	5040	3024	1680	840	360	120	24			
5	30240	15120	6720	2520	720	120				
6	151200	60480	20160	5040	720					
7	604800	181440	40320	5040						
8	1814400	362880	40320							
9	3628800	362880								
10	3628800									
S_p	9864100	986409	109600	13699	1956	325	64	15	4	1

The use of this table is obvious. Thus, the number of permutations of eight things in groups of five each is found in the fifth line of the column headed with the number 8. It will be noticed that the last two numbers in each column (excepting that headed with 1) are the same. This accords with the formula, which gives for the number of permutations of n things in groups of n the same value as for n things in groups of $(n - 1)$. It will also be remarked that the last number in each column of the table is the factorial, $n!$, of the number n at the head of the column. For example, in the column under 7, the last number is $5040 = 1 . 2 . 3 . 4 . 5 . 6 . 7 = 7!$.

The total number of permutations of n things taken singly, in groups of two, three, etc., is found by summing the numbers given by equation (1) for all values of r from 1 to n. Calling this total or sum S_p, it will be given by

$$S_p = \Sigma(n)_r. \tag{2}$$

To illustrate, suppose $n = 3$, and, to fix the ideas, let the three things be the three digits 1, 2, 3. Then from the above table it is seen that $S_p = 3 + 6 + 6 = 15$; or, that the number of numbers (all different) which can be formed from those digits is fifteen. These numbers are 1, 2, 3; 12, 13, 21, 23, 31, 32; 123, 132, 213, 231, 312, 321.

The values of S_p for $n = 1, 2, \ldots 10$ are given under the corresponding columns of the above table. But when n is large the direct summation indicated by (2) is tedious, if not impracticable. Hence a more convenient formula is desirable. To get this, observe that (1) may be written

$$(n)_r = \frac{n!}{(n-r)!}, \tag{1$'$}$$

if r is restricted to integer values between 1 and $(n - 1)$, both inclusive. This suffices to give all terms which appear in the right-hand member of (2), since the number of permutations for $r = (n - 1)$ is the same as for $r = n$. Hence it appears that

$$S_p = n! + \frac{n!}{1} + \frac{n!}{1 . 2} + \ldots \frac{n!}{(n-1)!}$$

$$= n!\left(1 + \frac{1}{1} + \frac{1}{1 . 2} + \ldots \frac{1}{(n-1)!}\right).$$

But as n increases, the series by which $n!$ is here multiplied approximates rapidly towards the base of natural logarithms; that is, towards

$$e = 2.7182818 +, \qquad \log e = 0.4342945.$$

Hence for large values of n

$$S_p = n!\,e, \text{ approximately.*} \qquad (3)$$

To get an idea of the degree of approximation of (3), suppose $n = 9$. Then the computation runs thus (see values in the above table):

	log
$9! = 362880$	5.5597630
e	0.4342945
$9!\,e = 986410$	5.9940575
$S_p = 986409$ by equation (2).	

The error in this case is thus seen to be only one unit, or about one-millionth of S_p.†

Prob. 1. Tabulate a list of the numbers of three figures each which can be formed from the first five digits 1, . . . 5. How many numbers can be formed from the nine digits?

Prob. 2. Is S_p always an odd number for n odd? Observe values of S_p in the table above.

ART. 3. COMBINATIONS.

In permutations attention is given to the order of arrangement of the things considered. In combinations no regard is paid to the order of arrangement. Thus, the permutations of the letters a, b, c, d in groups of three are

(abc)	(abd)	bac	bad	acb	(acd)	cab	cad
adb	adc	dab	dac	bca	(bcd)	cba	cbd
bda	bdc	dba	dbc	cda	cdb	dca	dcb

* See Art. 6 for a formula for computing $n!$ when n is a large number.

† When large numbers are to be dealt with, equations (1)′ and (3) are easily managed by logarithms, especially if a table of values of log ($n!$) is available. Such tables are given to six places in De Morgan's treatise on Probability in the Encyclopædia Metropolitana, and to five places in Shortrede's Tables (Vol. I, 1849).

But if the order of arrangement is ignored all of these are seen to be repetitions of the groups enclosed in parentheses, namely, (*abc*), (*abd*), (*acd*), (*bcd*). Hence in this case out of twenty-four permutations there are only four combinations.

A general formula for computing the number of combinations of n things taken in groups of r things is easily derived. For the number of permutations of n things in groups of r is by (1) of Art. 2

$$(n)_r = n(n-1)(n-2)\ldots(n-r+1);$$

and since each group of r things gives $1 \cdot 2 \cdot 3 \ldots r = r!$ permutations, the number of combinations must be the quotient of $(n)_r$ by $r!$. Denote this number by $C(n)_r$. Then the general formula is

$$C(n)_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \tag{1}$$

This formula gives, for example, in the case of the four letters a, b, c, d taken in groups of three, as considered above,

$$C(4)_3 = \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4.$$

Multiply both numerator and denominator of the right-hand member of (1) by $(n-r)!$ The result is

$$C(n)_r = \frac{n!}{r!\,(n-r)!}, \tag{1'}$$

which shows that the number of combinations of n things in groups of r is the same as the number of combinations of n things in groups of $(n-r)$. Thus, the number of combinations of the first ten letters $a, b, c \ldots j$ in groups of three or seven is

$$\frac{10!}{3!\,7!} = 120.$$

The following table gives the values $C(n)_r$ for all values of n and r from 1 to 10.

The mode of using this table is evident. For example, the number of combinations of eight things in sets of five each is found on the fifth line of the column headed 8 to be 56.

VALUES OF COMBINATIONS.

	10	9	8	7	6	5	4	3	2	1
1	10	9	8	7	6	5	4	3	2	1
2	45	36	28	21	15	10	6	3	1	
3	120	84	56	35	20	10	4	1		
4	210	126	70	35	15	5	1			
5	252	126	56	21	6	1				
6	210	84	28	7	1					
7	120	36	8	1						
8	45	9	1							
9	10	1								
10	1									
S_c	1023	511	255	127	63	31	15	7	3	

It will be observed that the numbers in any column show a maximum value when n is even and two equal maximum values when n is odd. That this should be so is easily seen from (1)′, which shows that $C(n)_r$ will be a maximum for any value of n when $r!\,(n-r)!$ is a minimum. For n even this is a minimum for $r = \frac{1}{2}n$; while for n odd it has equal minimum values for $r = \frac{1}{2}(n-1)$ and $r = \frac{1}{2}(n+1)$. Thus,

$$\text{maximum of } C(n)_r = \frac{n!}{\left(\frac{n}{2}!\right)^2} \text{ for } n \text{ even,}$$

$$= \frac{n!}{\frac{n+1}{2}!\,\frac{n-1}{2}!} \text{ for } n \text{ odd.} \qquad (2)$$

The total number of combinations of n things taken singly, in groups of two, three, etc., is found by summing the numbers given by (1) for all values of r from 1 to n both inclusive. Calling this total or sum S_c,

$$S_c = \Sigma C(n)_r.$$

The same sum will also come from (1)′ by giving to r all values from 1 to $(n-1)$, both inclusive, summing the results, and increasing their aggregate by unity. Thus by either process

$$S_c = n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-3)}{1 \cdot 2 \cdot 3} + \ldots + n + 1.$$

The second member of this equation is evidently equal to $(1 + 1)^n - 1$. Hence

$$S_c = \Sigma C(n)_r = 2^n - 1. \qquad (3)$$

The values of S_c for values of n and r from 1 to 10 are given under the corresponding columns of the above table.

Prob. 3. How many different squads of ten men each can be formed from a company of 100 men?

Prob. 4 How many triangles are formed by six straight lines each of which intersects the other five?

Prob. 5. Examine this statement: "In dealing a pack of cards the number of hands, of thirteen cards each, which can be produced is 635 013 559 600. But in whist four hands are simultaneously held, and the number of distinct deals . . . would require twenty-eight figures to express it." *

Prob. 6. Assuming combination always possible, and disregarding the question of proportions, find how many different substances could be produced by combining the seventy-three chemical elements.

ART. 4. DIRECT PROBABILITIES.

If it is known that one of two events must occur in any trial or instance, and that the first can occur in a ways and the second in b ways, all of which ways are equally likely to happen, then the probability that the first will happen is expressed mathematically by the fraction $a/(a + b)$, while the probability that the second will happen is $b/(a + b)$. Such events are said to be mutually exclusive. Denote their probabilities by p and q respectively. Then there result

$$p = \frac{a}{a + b}, \qquad q = \frac{b}{a + b}, \qquad p + q = 1, \qquad (1)$$

the last equation following from the first two and being the mathematical expression for the certainty that one of the two events must happen.

Thus, to illustrate, in tossing a coin it must give "head" or "tail"; $a = b = 1$, and $p = q = 1/2$. Again, if an urn contain $a = 5$ white and $b = 8$ black balls, the probability of drawing

* Jevons, Principles of Science, p. 217.

a white ball in one trial is $p = 5/13$ and that of drawing a black one $q = 8/13$.

Similarly, if there are several mutually exclusive events which can occur in $a, b, c \ldots$ ways respectively, their probabilities $p, q, r \ldots$ are given by

$$p = \frac{a}{a+b+c+\ldots}, \quad q = \frac{b}{a+b+c+\ldots}, \quad r = \frac{c}{a+b+c+\ldots}.$$

$$p + q + r + \ldots = 1. \qquad (2)$$

For example, if an urn contain $a = 4$ white, $b = 5$ black, and $c = 6$ red balls, the probabilities of drawing a white, black, and red ball at a single trial are $p = 4/15$, $q = 5/15$, and $r = 6/15$, respectively.

Formulas (1) and (2) may be applied to a wide variety of cases, but it must suffice here to give only a few such. As a first illustration, consider the probability of drawing at random a number of three figures from the entire list of numbers which can be formed from the first seven digits. A glance at the table of Art. 1 shows that the symbols of formula (1) have in this case the values $a = 210$, and $a + b = 13699$. Hence $b = 13489$, and $p = 210/13699$; that is, the probability in question is about $1/65$.

Secondly, what is the probability of holding in a hand of whist all the cards of one suit? Formula (1) of Art. 3 shows that the number of different hands of thirteen cards each which may be formed from a pack of fifty-two cards is

$$\frac{52 \cdot 51 \cdot 50 \ldots 40}{1 \cdot 2 \cdot 3 \ldots 13} = 635\,013\,559\,600,$$

and the probability required is the reciprocal of this number. The probability against this event is, therefore, very nearly unity.

Thirdly, consider the probabilities presented by the case of an urn containing 4 white, 5 black, and 6 red balls, from which at a single trial three balls are to be drawn. Evidently the triad of balls drawn may be all white, all black, all red, partly white and black, partly white and red, partly black and red, or one each of the white, black, and red. There are thus seven different probabilities to be taken into account. The theory of combinations shows (see equation (1), Art. 3) that the total number of

White triads	$= \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}$	$= \quad 4 = a$
Black triads	$= \frac{5 \cdot 4 \cdot 3}{6}$	$= 10 = b$
Red triads	$= \frac{6 \cdot 5 \cdot 4}{6}$	$= 20 = c$
White and black triads	$= \frac{9 \cdot 8 \cdot 7}{6} - (4+10)$	$= 70 = d$
White and red triads	$= \frac{10 \cdot 9 \cdot 8}{6} - (4+20)$	$= 96 = e$
Black and red triads	$= \frac{11 \cdot 10 \cdot 9}{6} - (10+20)$	$= 135 = f$
White, black, and red triads	$= 4 \cdot 5 \cdot 6$	$= 120 = g$
	Sum	$= 455$

The total number of these triads is 455, and is, as it should be, the number of combinations in groups of three each of the whole number of balls. Hence formulas (2) give the seven different probabilities which follow, using the initial letters w, b, r to indicate the colors represented in a triad :

For a triad *www*	$p = 4/455$,
" " " *bbb*	$q = 10/455$,
" " " *rrr*	$r = 20/455$,
" " " *wwb* or *wbb*	$s = 70/455$,
" " " *wwr* or *wrr*	$t = 96/455$,
" " " *bbr* or *brr*	$u = 135/455$,
" " " *wbr*	$v = 120/455$.

Prob. 7. When three dice are thrown together, what is the probability that the throw will be greater than 9 ?

Prob. 8. Write down a literal formula for the probabilities of the several possible triads considered in the above question of the balls, supposing the numbers of white, black, and red balls to be l, m, n, respectively.

ART. 5. PROBABILITY OF CONCURRENT EVENTS.

If the probabilities of two independent events are p_1 and p_2, respectively, the probability of their concurrence in any single instance is p_1p_2. Thus, suppose there are two urns U_1 and U_2, the first of which contains a_1 white and b_1 black balls, and the second a_2 white and b_2 black balls. Then the probability of drawing a white ball from U_1 is $p_1 = a_1/(a_1 + b_1)$, while that of drawing a white ball from U_2 is $p_2 = a_2/(a_2 + b_2)$. The total number of different pairs of balls which can be formed from the entire number of balls is $(a_1 + b_1)(a_2 + b_2)$. Of these pairs a_1a_2 are favorable to the concurrence of white in simultaneous or successive drawings from the two urns. Hence the probability of a concurrence of

white with white $= a_1a_2/(a_1 + b_1)(a_2 + b_2)$,
white with black $= (a_1b_2 + a_2b_1)/(a_1 + b_1)(a_2 + b_2)$,
black with black $= b_1b_2/(a_1 + b_1)(a_2 + b_2)$,

and the sum of these is unity, as required by equations (2) of Article 4.

In general, if $p_1, p_2, p_3 \ldots$ denote the probabilities of several independent events, and P denote the probability of their concurrence,

$$P = p_1p_2p_3. \tag{1}$$

To illustrate this formula, suppose there is required the probability of getting three aces with three dice thrown simultaneously. In this case $p_1 = p_2 = p_3 = 1/6$ and $P = (1/6)^3 = 1/216$.

Similarly, if two dice are thrown simultaneously the probability that the sum of the numbers shown will be 11 is $2/36$; and the probability that this sum 11 will appear in two successive throws of the same pair of dice is $4/36.36$.

The probability that the alternatives of a series of events will concur is evidently given by

$$Q = q_1q_2q_3 \ldots = (1 - p_1)(1 - p_2)(1 - p_3) \tag{2}$$

Thus, in the case of the three dice mentioned above, the probability that each will show something other than an ace is

$q_1 = q_2 = q_3 = 5/6$, and the probability that they will concur in this is $Q = 125/216$.

Many cases of interest occur for the application of (1) and (2). One of the most important of these is furnished by successive trials of the same event. Consider, for example, what may happen in n trials of an event for which the probability is p and against which the probability is q. The probability that the event will occur every time is p^n. The probability that the event will occur $(n - 1)$ times in succession and then fail is $p^{n-1}q$. But if the order of occurrence is disregarded this last combination may arrive in n different ways; so that the probability that the event will occur $(n - 1)$ times and fail once is $np^{n-1}q$. Similarly, the probability that the event will happen $(n - 2)$ times and fail twice is $\frac{1}{2}n(n - 1)p^{-n2}q^2$; etc. That is, the probabilities of the several possible occurrences are given by the corresponding terms in the development of $(p + q)^n$.

By the same reasoning used to get equations (2) of Art. 3 it may be shown that the maximum term in the expansion of $(p + q)^n$ is that in which the exponent m, say, of q is the whole number lying between $(n + 1)q - 1$ and $(n + 1)q$. In other words, the most probable result in n trials is the occurrence of the event $(n - m)$ times and its failure m times. When n is large this means that the most probable of all possible results is that in which the event occurs $n - nq = n(1 - q) = np$ times and fails nq times. Thus, if the event be that of throwing an ace with a single die the most probable of the possible results in 600 throws is that of 100 aces and 500 failures.

Since q^n is the probability that the event will fail every time in n trials, the probability that it will occur at least once in n trials is $1 - q^n$. Calling this probability r,*

$$r = 1 - q^n = 1 - (1 - p)^n. \tag{3}$$

If r in this equation be replaced by $1/2$, the corresponding value of n is the number of trials essential to render the

* See Poisson's Probabilité des Jugements, pp. 40, 41.

chances even that the event whose probability is p will occur at least once. Thus, in this case, the value of n is given by

$$n = -\frac{\log 2}{\log(1-p)}.$$

This shows, for example, if the event be the throwing of double sixes with two dice, for which $p = 1/36$, that the chances are even ($r = 1/2$) that in 25 throws ($n = 24.614$ by the formula) double sixes will appear at least once.

Equation (3) shows that however small p may be, so long as it is finite, n may be taken so large as to make r approach indefinitely near to unity; that is, n may be so large as to render it practically certain that the event will occur at least once.

When n is large

$$\begin{aligned}(1-p)^n &= 1 - np + \frac{n(n-1)}{1.2}p^2 - \frac{n(n-1)(n-2)}{1.2.3}p^3 + \dots \\ &= 1 - np + \frac{(np)^2}{1.2} - \frac{(np)^3}{1.2.3} + \dots \\ &= e^{-np} \text{ approximately.}\end{aligned}$$

Thus an approximate value of r is

$$r = 1 - e^{-np}, \qquad \log e = 0.4342495. \tag{4}$$

This formula gives, for example, for the probability of drawing the ace of spades from a pack of fifty-two cards at least once in 104 trials $r = 1 - e^{-2} = 0.865$, while the exact formula (3) gives 0.867.

Similarly, the probability of the occurrence of the event at least t times in n trials will be given by the sum of the terms of $(p+q)^n$ from p^n up to that in $p^t q^{n-t}$ inclusive. This probability must be carefully distinguished from the probability that the event will occur t times only in the n trials, the latter being expressed by the single term in $p^t q^{n-t}$.

Prob. 9. Compare the probability of holding exactly four aces in five hands of whist with the probability of holding at least four aces in the same number of hands.

Prob. 10. What is the probability of an event if the chances are even that it occurs at least once in a million trials? See equation (4).

ART. 6. BERNOULLI'S THEOREM.

Denote the exponents of p and q in the maximum term of $(p+q)^n$ by μ and m respectively, and denote this term by T. Then

$$T = \frac{n(n-1)(n-2)\ldots(\mu+1)}{m!}p^\mu q^m = \frac{n!}{\mu!\,m!}p^\mu q^m. \quad (1)$$

As shown in Art. 5, μ in this formula is the greatest whole number in $(n+1)p$, and m the greatest whole number in $(n+1)q$; so that when n is large, μ and m are sensibly equal to np and nq respectively.

The direct calculation of T by (1) is impracticable when n is large. To overcome this difficulty the following expression is used: *

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right). \quad (2)$$

$$\log e = 0.4342495, \quad \log 2\pi = 0.7981799.$$

This expression approaches $n^n e^{-n}\sqrt{2\pi n}$ as a limit with the increase of n, and in this approximate form is known as Stirling's theorem. Although a rude approximation to $n!$ for small values of n this theorem suffices in nearly all cases wherein such probabilities as T are desired. Making use of the theorem in (1) it becomes

$$T = \frac{1}{\sqrt{2\pi npq}}. \quad (3)$$

That this formula affords a fair approximation even when n is small is seen from the case of a die thrown 12 times. The probability that any particular face will appear in one throw is $p = 1/6$, whence $q = 5/6$; and the most probable result in 12 throws is that in which the particular face appears twice and fails to appear ten times. The probability of this result computed from (3) is 0.309, while the exact formula (1) gives 0.296.

The probability that the event will occur a number of times

* This expression is due to Laplace, Théorie Analytique des Probabilités. See also De Morgan's Calculus, pp. 600–604.

comprised between $(\mu - \alpha)$ and $(\mu + \alpha)$ in n trials is evidently expressed by the sum of the terms in $(p + q)^n$ for which the exponent of p has the specified range of values. Calling this probability R, putting

$$\mu = np + u, \quad \text{and} \quad m = nq - u,$$

and using Stirling's theorem (which implies that n is a large number),*

$$R = \Sigma \frac{1}{\sqrt{2\pi npq}} \left(1 + \frac{u}{np}\right)^{-(np+u)} \left(1 - \frac{u}{nq}\right)^{-(nq-u)},$$

very nearly; and the summation is with respect to u from $u = -\alpha$ to $u = +\alpha$. But expansion shows that the natural logarithm of the product of the two binomial factors in this equation is approximately $-u^2/2npq$. Hence

$$R = \Sigma \frac{1}{\sqrt{2\pi npq}} e^{-u^2/2npq};$$

and, since n is supposed large, this may be replaced by a definite integral, putting

$$dz = 1/\sqrt{2npq}, \quad \text{and} \quad z^2 = u^2/2npq.$$

Thus

$$R = \frac{1}{\sqrt{\pi}} \int_{-\alpha/\sqrt{2npq}}^{+\alpha/\sqrt{2npq}} e^{-z^2} dz = \frac{2}{\sqrt{\pi}} \int_0^{\alpha/\sqrt{2npq}} e^{-z^2} dz. \tag{4}$$

This equation expresses the theorem of James Bernoulli, given in his Ars Conjectandi, published in 1713.

The value of the right-hand member of (4) varies, as it should, between 0 and 1, and approaches the latter limit rapidly as z increases. Thus, writing for brevity

$$I = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz,$$

* See Bertrand, Calcul des Probabilités, Paris, 1889, for an extended discussion of the questions considered in this Article.

the following table shows the march of the integral:

z	I	z	I	z	I
0.00	0.000	0.75	0.711	1.50	0.966
.25	.276	1.00	.843	1.75	.987
.50	.520	1.25	.923	2.00	.995

To illustrate the use of (4), suppose there is required the probability that in 6000 throws of a die the ace will appear a number of times which shall be greater than $1/6 \times 6000 - 10$ and less than $1/6 \times 6000 + 10$, or a number of times lying between 990 and 1010. In this case $\alpha = 10$, $n = 6000$, $p = 1/6$, $q = 5/6$. Thus, $\alpha/\sqrt{2npq} = 10/\sqrt{2 \,.\, 6000 \,.\, 1/6 \,.\, 5/6} = 0.245$. Hence, by (4) and the table, $R = 0.27$.

Prob. 11. If the ratio of males to females at birth is 105 to 100, what is the probability that in the next 10,000 births the number of males will fall within two per cent of the most probable number?

Prob. 12. If the chance is even for head and tail in tossing a coin, what is the probability that in a million throws the difference between heads and tails will exceed 1500?

ART. 7. INVERSE PROBABILITIES.*

If an observed event can be attributed to any one of several causes, what is the probability that any particular one of these causes produced the event? To put the question in a concrete form, suppose a white ball has been drawn from one of two urns, U_1 containing 3 white and 5 black balls, and U_2 containing 2 white and 4 black balls; and that the probability in favor of each urn is required. If U_1 is as likely to have been chosen as U_2, the probability that U_1 was chosen is $1/2$. After such choice the probability of drawing a white ball from U_1 is $3/8$. Before drawing, therefore, the probability of getting a white ball from U_1 was $1/2 \times 3/8 = 3/16$, by Art. 5. Similarly, before drawing the probability of getting a white ball from U_2 was $1/2 \times 2/6 = 1/6$. These probabilities will remain unchanged if the number of balls in either urn be increased or

* See Poisson, Probabilité des Jugements, pp. 81–83.

diminished so long as the ratio of white to black balls is kept constant. Make these numbers the same for the two urns. Thus let the first contain 9 white and 15 black, and the second 8 white and 16 black; whence the above probabilities may be written $1/2 \times 9/24$ and $1/2 \times 8/24$. It is now seen that there are $(9 + 8)$ cases favorable to the production of a white ball, each of which has the same antecedent probability, namely, $1/2$. Since the fact that a white ball was drawn excludes consideration of the black balls, the probability that the white ball came from U_1 is $9/17$ and that it came from U_2 is $8/17$; and the sum of these is unity, as it should be.

To generalize this result, let there be m causes, $C_1, C_2, \ldots C_m$. Denote their direct probabilities by $q_1, q_2, \ldots q_m$; their antecedent probabilities by $r_1, r_2, \ldots r_m$; and their resultant probabilities on the supposition of separate existence by $p_1, p_2, \ldots p_m$. That is,

$$p_1 = q_1 r_1, \quad p_2 = q_2 r_2, \ldots p_m = q_m r_m. \tag{1}$$

Let D be the common denominator of the right-hand members in (1), and denote the corresponding numerators of the several fractions by $s_1, s_2, \ldots s_m$. Then

$$p_1 = s_1/D, \quad p_2 = s_2/D, \ldots p_m = s_m/D;$$

and it is seen that there are in all $(s_1 + s_2 + \ldots s_m)$ equally possible cases, and that of these s_1 are favorable to C_1, s_2 to $C_2, \ldots$ Hence, if $P_1, P_2, \ldots P_m$ denote the probabilities of the several causes on the supposition of their coexistence,

$$P_1 = s_1/(s_1 + s_2 + \ldots s_m) = p_1/(p_1 + p_2 + \ldots p_m).$$

Thus in general

$$P_1 = p_1/\Sigma p, \quad P_2 = p_2/\Sigma p, \ldots P_m = p_m/\Sigma p. \tag{2}$$

To illustrate the meaning of these formulas by the above concrete case of the urns it suffices to observe that

for U_1, $q_1 = 3/8$ and $r_1 = 1/2$,
for U_2, $q_2 = 1/3$ and $r_2 = 1/2$;
whence $p_1 = 3/16$, $p_2 = 1/6$, $p_1 + p_2 = 17/48$;
and $P_1 = 9/17$, $P_2 = 8/17$.

As a second illustration, suppose it is known that a white ball has been drawn from an urn which originally contained m balls, some of them being black, if all are not white. What is the probability that the urn contained exactly n white balls? The facts are consistent with m different and equally probable hypotheses (or causes), namely, that there were 1 white and $(m-1)$ black balls, 2 white and $(m-2)$ black balls, etc. Hence in (1), $q_1 = q_2 = \ldots = 1$, and

$$p_1 = 1/m, \quad p_2 = 2/m, \ldots p_n = n/m, \ldots p_m = m/m.$$

Thus
$$\Sigma p = (1/2)(m+1),$$

and
$$P_n = p_n/\Sigma p = \frac{2n}{m(m+1)}.$$

This shows, as it evidently should, that $n = m$ is the most probable number of white balls in the urn. The probability for this number is $P_m = 2/(m+1)$, which reduces, as it ought, to 1 for $m = 1$.

Formulas (1) and (2) may also be applied to the problem of estimating the probability of the occurrence of an event from the concurrent testimony of several witnesses, $X_1, X_2, \ldots$ Denote the probabilities that the witnesses tell the truth by x_1 $x_2, \ldots$ Then, supposing them to testify independently, the probability that they will concur in the truth concerning the event is $x_1x_2 \ldots$; while the probability that they will concur in the only other alternative, falsehood, is $(1-x_1)(1-x_2)\ldots$ The two alternatives are equally possible. Hence by equations (1) and (2)

$$\begin{aligned} p_1 &= x_1x_2 \ldots, \quad p_2 = (1-x_1)(1-x_2)\ldots, \\ P_1 &= \frac{x_1x_2\ldots}{x_1x_2\ldots + (1-x_1)(1-x_2)\ldots}, \\ P_2 &= \frac{(1-x_1)(1-x_2)\ldots}{x_1x_2\ldots + (1-x_1)(1-x_2)\ldots}, \end{aligned} \qquad (3)$$

P_1 being the probability for and P_2 that against the event.

To illustrate (3), if the chances are 3 to 1 that X_1 tells the truth and 5 to 1 that X_2 tells the truth, $x_1 = 3/4$, $x_2 = 5/6$, and $P_1 = 15/16$; or, the chances are 15 to 1 that an event occurred if they agree in asserting that it did.*

* For some interesting applications of equations (3) see note E of Appendix to the Ninth Bridgewater Treatise by Charles Babbage (London, 1838).

It is of theoretical interest to observe that if x_1, x_2, . . . in (3) are each greater than 1/2, P_1 approaches unity as the number of witnesses is indefinitely increased.

Prob. 13. The groups of numbers of one figure each, two figures each, three figures each, etc., which it is possible to form from the nine digits 1, 2, . . . 9 are printed on cards and placed severally in nine similar urns. What is the probability that the number 777 will be drawn in a single trial by a person unaware of the contents of the urns?

Prob. 14. How many witnesses whose credibilities are each 3/4 are essential to make $P_1 = 0.999$ in equation (3)?

ART. 8. PROBABILITIES OF FUTURE EVENTS.

Equations (2) of Art. 7 may be written in the following manner:

$$\frac{P_1}{p_1} = \frac{P_2}{p_2} = \ldots \frac{P_m}{p_m} = \frac{1}{\Sigma p}. \qquad (1)$$

If $p_1, p_2, \ldots p_m$ are found by observation, $P_1, P_2, \ldots P_m$ will express the probabilities of the corresponding causes or their effects. When, as in the case of most physical facts, the number of causes and events is indefinitely great, the value of any p or P in (1) becomes indefinitely small, and the value of Σp must be expressed by means of a definite integral. Let x denote the probability of any particular cause, or of the event to which it gives rise. Then, supposing this and all the other causes mutually exclusive, $(1 - x)$ will be the probability against the event. Now suppose it has been observed that in $(m + n)$ cases the event in question has occurred m times and failed n times. The probability of such a concurrence is, by Art. 5, $cx^m(1 - x)^n$, where c is a constant. Since x is unknown, it may be assumed to have any value within the limits 0 and 1; and all such values are à priori equally possible. Put

$$y = cx^m(1 - x)^n.$$

Then evidently the probability that x will fall within any assigned possible limits a and b is expressed by the fraction

$$\int_a^b y\,dx \Big/ \int_0^1 y\,dx;$$

so that the probability of any particular x is given by

$$P = \frac{x^m(1-x)^n dx}{\int_0^1 x^m(1-x)^n dx}. \qquad (2)$$

This may be regarded as the antecedent probability of the cause or event in question.

What then is the probability that in the next $(r+s)$ trials the event will occur r times and fail s times, if no regard is had of the order of occurrence? If x were known, the answer would be by Arts. 2 and 5

$$\frac{(r+s)!}{r!\,s!} x^r(1-x)^s. \qquad (3)$$

But since x is restricted only by the condition (2), the required probability will be found by taking the product of (2) and (3) and integrating throughout the range of x. Thus, calling the required probability Q,

$$Q = \frac{(r+s)!}{r!\,s!} \frac{\int_0^1 x^{m+r}(1-x)^{n+s} dx}{\int_0^1 x^m(1-x)^n dx}. \qquad (4)$$

The definite integrals which appear here are known as Gamma functions. They are discussed in all of the higher treatises on the Integral Calculus. Applying the rules derived in such treatises there results *

$$Q = \frac{(r+s)!\,(m+r)!\,(n+s)!\,(m+n+1)!}{r!\,s!\,m!\,n!\,(m+n+r+s+1)!}. \qquad (5)$$

If regard is had to the order of occurrence of the event; that is, if the probability required is that of the event happening r times in succession and then failing s times in succession,

* It is a remarkable fact that formula (5) is true without restriction as to values of m, n, r, s. The formula may be established by elementary considerations, as was done by Prevost and Lhuilier, 1795. See Todhunter's History of he Theory of Probability, pp. 453–457.

the factor $(r+s)!/r!s!$ in (3), (4), (5) must be replaced by unity.

To illustrate these formulas, suppose first that the event has happened m times and failed no times. What is the probability that it will occur at the next trial? In this case (4) gives

$$Q = \int_0^1 x^{m+1}dx \Big/ \int_0^1 x^m dx = (m+1)/(m+2).$$

When m is large this probability is nearly unity. Thus, the sun has risen without failure a great number of times m; the probability that it will rise to-morrow is

$$\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)^{-1} = 1 + \frac{1}{m} - \frac{2}{m} + \ldots$$

which is practically 1.

Secondly, suppose an urn contains white and black balls in an unknown ratio. If in ten trials 7 white and 3 black balls are drawn, what is the probability that in the next five trials 2 white and 3 black balls will be drawn? The application of (5) supposes the ratio of the white and black balls in the urn to remain constant. This will follow if the balls are replaced after each drawing, or if the number of balls in the urn is supposed infinite. The data give

$$m = 7, \quad n = 3, \quad r = 2, \quad s = 3,$$
$$m + r = 9, \quad n + s = 6, \quad r + s = 5, \quad m + n + 1 = 11,$$
$$m + n + r + s + 1 = 16.$$

Thus by (5)

$$Q = \frac{5!9!6!11!}{2!3!7!3!16!} = 30/91.$$

Suppose there are two mutually exclusive events, the first of which has happened m times and the second n times in $m+n$ trials. What is the probability that the chance of the occurrence of the first exceeds $1/2$? The answer to this question is given directly by equation (2) by integrating the numerator between the specified limits of x. That is,

$$P = \frac{\int_{0.5}^{1} x^m(1-x)^n dx}{\int_{0}^{1} x^m(1-x)^n dx}. \quad (6)$$

Thus, if $m = 1$ and $n = 0$, $P = 3/4$; or the odds are three to one that the event is more likely to happen than not. Similarly, if the event has occurred m times in succession,

$$P = 1 - (1/2)^{m+1},$$

which approaches unity rapidly with increase of n.

ART. 9. THEORY OF ERRORS.

The theory of errors may be defined as that branch of mathematics which is concerned, first, with the expression of the resultant effect of one or more sources of error to which computed and observed quantities are subject; and, secondly, with the determination of the relation between the magnitude of an error and the probability of its occurrence. In the case of computed quantities which depend on numerical data, such as tables of logarithms, trigonometric functions, etc., it is usually possible to ascertain the actual values of the resultant errors. In the case of observed quantities, on the other hand, it is not generally possible to evaluate the resultant actual error, since the actual errors of observation are usually unknown. In either case, however, it is always possible to write down a symbolical expression which will show how different sources of error enter and affect the aggregate error; and the statement of such an expression is of fundamental importance in the theory of errors.

To fix the ideas, suppose a quantity Q to be a function of several independent quantities $x, y, z \ldots$; that is,

$$Q = f(x, y, z \ldots),$$

and let it be required to determine the error in Q due to errors in $x, y, z \ldots$ Denote such errors by $\Delta Q, \Delta x, \Delta y, \Delta z \ldots$ Then, supposing the errors so small that their squares, products, and higher powers may be neglected, Taylor's series gives

$$\Delta Q = \frac{\partial Q}{\partial x}\Delta x + \frac{\partial Q}{\partial y}\Delta y + \frac{\partial Q}{\partial z}\Delta z + \ldots \qquad (1)$$

This equation may be said to express the resultant actual error of the function in terms of the component actual errors, since the actual value of ΔQ is known when the actual errors of $x, y, z \ldots$ are known. It should be carefully noted that the quantities $x, y, z \ldots$ are supposed subject to errors which are independent of one another. The discovery of the independent sources of error is sometimes a matter of difficulty, and in general requires close attention on the part of the student if he would avoid blunders and misconceptions. Every investigator in work of precision should have a clear notion of the error-equation of the type (1) appertaining to his work; for it is thus only that he can distinguish between the important and unimportant sources of error.

Prob. 15. Write out the error-equation in accordance with (1) for the function $Q = xyz + x^3 \log (y/z)$.

Prob. 16. In a plane triangle $a/b = \sin A/\sin B$. Find the error in a due to errors in b, A, and B.

Prob. 17. Suppose in place of the data of problem 16 that the angles used in computation are given by the following equations: $A = A_1 + \frac{1}{3}(180° - A_1 - B_1 - C_1)$, $B = B_1 + \frac{1}{3}(180° - A_1 - B_1 - C_1)$, where A_1, B_1, C_1 are observed values. What then is Δa?

Prob. 18. If w denote the weight of a body and r the radius of the earth, show that for small changes in altitude, $\Delta w/w = - \Delta r/r$; whence, if a precision of one part in 500 000 000 is attainable in comparing two nearly equal masses, the effect of a difference in altitude of one centimeter in the scale-pans of a balance will be noticeable.*

ART. 10. LAWS OF ERROR.

A law of error is a function which expresses the relative frequency of occurrence of errors in terms of their magnitudes. Thus, using the customary notation, let ϵ denote the magni-

* This problem arose with the International Bureau of Weights and Measures, whose work of intercomparison of the Prototype Kilogrammes attained a precision indicated by a probable error of 1/500 000 000th part of a kilogramme.

tude o. any error in a system of possible errors. Then the law of such system may be expressed by an equation of the form

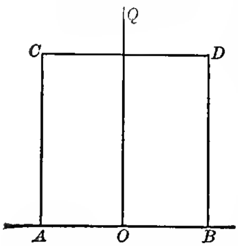
$$y = \phi(\epsilon). \qquad (1)$$

Representing ϵ as abscissa and y as ordinate, this equation gives a curve called the curve of frequency, the nature of which, as is evident, depends on the form of the function ϕ. This equation gives the relative frequency of occurrence of errors in the system; so that if ϵ is continuous the probability of the occurrence of any particular error is expressed by $yd\epsilon = \phi(\epsilon)d\epsilon$; which is infinitesimal, as it plainly should be, since in any continuous system the number of different values of ϵ is infinite.

Consider the simplest form of $\phi(\epsilon)$, namely, that in which $\phi(\epsilon) = c$, a constant. This form of $\phi(\epsilon)$ obtains in the case of the errors of tabular logarithms, natural trigonometric functions, etc. In this case all errors between minus a half-unit and plus a half-unit of the last tabular place are equally likely to occur. Suppose, to cover the class of cases to which that just cited belongs, all errors between the limits $-a$ and $+a$ are equally likely to occur. The probability of any individual error will then be $\phi(\epsilon)d\epsilon = cd\epsilon$, and the sum of all such probabilities, by equation (2), Art. 4, must be unity. That is,

$$\int_{-a}^{+a} \phi(\epsilon)d\epsilon = c\int_{-a}^{+a} d\epsilon = 1.$$

This gives $c = 1/2a$, or by (1) $y = 1/2a$. The curve of frequency in this case is shown in the figure, AB being the axis of ϵ and OQ that of y. It is evident from this diagram that if the errors of the system be considered with respect to magnitude only, half of them should be greater and half less than $a/2$. This is easily found to be so in the case of tabular logarithms, etc.



As a second illustration of (1), suppose y and ϵ connected by the relation $y = c\sqrt{a^2 - \epsilon^2}$, where a is the radius of a circle,

c a constant, and ϵ may have any value between $-a$ and $+a$. Then the condition

$$c\int_{-a}^{+a} d\epsilon \sqrt{a^2 - \epsilon^2} = 1$$

gives $c = 2/(a^2\pi)$. In this, as in the preceding case, $\phi(+\epsilon) = \phi(-\epsilon)$, the meaning of which is that positive and negative errors of the same magnitude are equally likely to occur. It will be noticed, however, that in the latter case small errors have a much higher probability than those near the limit a, while in the former case all errors have the same probability.

In general, when ϵ is continuous $\phi(\epsilon)$ must satisfy the condition $\int \phi(\epsilon) d\epsilon = 1$, the limits being such as to cover the entire range of values of ϵ. The cases most commonly met with are those in which $\phi(\epsilon)$ is an even function, or those in which $\phi(+\epsilon) = \phi(-\epsilon)$. In such cases, if $\pm a$ denote the limiting value of ϵ,

$$\int_{-a}^{+a} \phi(\epsilon) d\epsilon = 2\int_{0}^{a} \phi(\epsilon) d\epsilon = 1. \qquad (3)$$

ART. 11. TYPICAL ERRORS OF A SYSTEM.

Certain typical errors of a system have received special designations and are of constant use in the literature of the theory of errors. These special errors are the probable error, the mean error, and the average error. The first is that error of the system of errors which is as likely to be exceeded as not; the second is the square root of the mean of the squares of all the errors; and the third is the mean of all the errors regardless of their signs. Confining attention to systems in which positive and negative errors of the same magnitude are equally probable, these typical errors are defined mathematically as follows. Let

ϵ_p = the probable error,

ϵ_m = the mean error,

ϵ_a = the average error.

Then, observing (2), of Art. 10,

$$\left.\begin{aligned}\int_{-a}^{-\epsilon_p}\phi(\epsilon)d\epsilon = \int_{-\epsilon_p}^{0}\phi(\epsilon)d\epsilon = \int_{0}^{+\epsilon_p}\phi(\epsilon)d\epsilon = \int_{+\epsilon_p}^{+a}\phi(\epsilon)d\epsilon = \tfrac{1}{4}.\\ \epsilon_m{}^2 = \int_{-a}^{+a}\phi(\epsilon)\epsilon^2 d\epsilon, \quad \epsilon_a = 2\int_{0}^{+a}\phi(\epsilon)\epsilon d\epsilon.\end{aligned}\right\} \quad (1)$$

The student should seek to avoid the very common misapprehension of the meaning of the probable error. It is not "the most probable error," nor "the most probable value of the actual error"; but it is that error which, disregarding signs, would occupy the middle place if all the errors of the system were arranged in order of magnitude. A few illustrations will suffice to fix the ideas as to the typical errors. Thus, take the simple case wherein $\phi(\epsilon) = c = 1/2a$, which applies to tabular logarithms, etc. Equations (1) give at once

$$\epsilon_p = \pm \frac{1}{2}a, \quad \epsilon_m = \pm \frac{a}{3}\sqrt{3}, \quad \epsilon_a = \pm \frac{1}{2}a.$$

For the case of tabular values, $a = 0.5$ in units of the last tabular place. Hence for such values

$$\epsilon_p = \pm 0.25, \quad \epsilon_m = \pm 0.29, \quad \epsilon_a = \pm 0.25.$$

Prob. 19. Find the typical errors for the cases in which the law of error is $\phi(\epsilon) = c\sqrt{a^2 - \epsilon^2}$, $\phi(\epsilon) = c(\pm a \mp \epsilon)$, $\phi(\epsilon) = c\cos^2(\pi\epsilon/2a)$; c being a constant to be determined in each case and ϵ having any value between $-a$ and $+a$.

ART. 12. LAWS OF RESULTANT ERROR.

When several independent sources of error conspire to produce a resultant error, as specified by equation (1) of Art. 9, there is presented the problem of determining the law of the resultant error by means of the laws of the component errors. The algebraic statement of this problem is obtained as follows for the case of continuous errors:

In the equation (1), Art. 9, write for brevity

$$\epsilon = \Delta Q, \quad \epsilon_1 = \frac{\partial Q}{\partial x}\Delta x, \quad \epsilon_2 = \frac{\partial Q}{\partial y}\Delta y, \ldots;$$

and let the laws of error of ϵ, ϵ_1, ϵ_2, ... be denoted by $\phi(\epsilon)$, $\phi_1(\epsilon_1)$, $\phi_2(\epsilon_2)$ ... Then the value of ϵ is given by

$$\epsilon = \epsilon_1 + \epsilon_2 + \dots \qquad (1)$$

The probabilities of the occurrence of any particular values of ϵ_1, ϵ_2, ... are given by $\phi_1(\epsilon_1)d\epsilon_1$, $\phi_2(\epsilon_2)d\epsilon_2$, ... ; and the probability of their concurrence is the probability of the corresponding value of ϵ. But since this value may arise in an infinite number of ways through the variations of ϵ_1, ϵ_2, ... over their ranges, the probability of ϵ, or $\phi(\epsilon)d\epsilon$, will be expressed by the integral of $\phi_1(\epsilon_1)d\epsilon_1\phi_2(\epsilon_2)d\epsilon_2$ ... subject to the restriction (1). This latter gives $\epsilon_1 = \epsilon - \epsilon_2 - \epsilon_3$ ..., and $d\epsilon_1 = d\epsilon$ for the multiple integration with respect to ϵ_2, ϵ_3, ... Hence there results

$$\phi(\epsilon)d\epsilon = d\epsilon\int\phi_1(\epsilon - \epsilon_2 - \epsilon_3 - \dots)\phi_2(\epsilon_2)d\epsilon_2 \dots,$$

or

$$\phi(\epsilon) = \int\phi_1(\epsilon - \epsilon_1 - \epsilon_2 - \dots)\phi_2(\epsilon_2)d\epsilon_2\int\phi_3(\epsilon_3)d\epsilon_3 \dots \qquad (2)$$

It is readily seen that this formula will increase rapidly in complexity with the number of independent sources of error.* For some of the most important practical applications, however, it suffices to limit equation (2) to the case of two independent sources of error, each of constant probability within assigned limits. Thus, to consider this case, let ϵ_1 vary over the range $-a$ to $+a$, and ϵ_2 vary over the range $-b$ to $+b$. Then by equation (2), Art. 10,

$$\phi_1(\epsilon_1) = 1/(2a), \quad \phi_2(\epsilon_2) = 1/(2b).$$

Hence equation (2) becomes

$$\phi(\epsilon) = \frac{1}{4ab}\int d\epsilon_2.$$

In evaluating this integral ϵ_2 must not surpass $\pm b$ and $\epsilon_1 = \epsilon - \epsilon_2$ must not surpass $\pm a$. Assuming $a > b$, the limits of the integral for any value of $\epsilon = \epsilon_1 + \epsilon_2$ lying between $-(a + b)$ and $-(a - b)$ are $-b$ and $+(\epsilon + a)$. This fact is

* The reader desirous of pursuing this phase of the subject should consult Bessel's Untersuchungen ueber die Wahrscheinlichkeit der Beobachtungsfehler; Abhandlungen von Bessel (Leipzig, 1876), Vol. II.

made plain by a numerical example. For instance, suppose $a = 5$ and $b = 3$. Then $-(a+b) = -8$ and $-(a-b) = -3$. Take $\epsilon = -6$, a number intermediate to -8 and -3. Then the following are the possible integer values of ϵ_1 and ϵ_2 which will produce $\epsilon = -6$:

ϵ	ϵ_1	ϵ_2	limits of ϵ_2
$-6 =$	-5	$-1,$	$-1 = +(\epsilon + a),$
$=$	-4	$-2,$	
$=$	-3	$-3,$	$-3 = -b.$

Similarly, the limits of ϵ_2 for values of ϵ lying between $-(a-b)$ and $+(a-b)$ are $-b$ and $+b$; and the limits of ϵ_2 for values of ϵ between $+(a-b)$ and $+(a+b)$ are $+(\epsilon - a)$ and $+b$. Hence

$$\left.\begin{aligned}
\phi(\epsilon) &= \frac{1}{4ab}\int_{-b}^{\epsilon+a} d\epsilon_2 = \frac{\epsilon+a+b}{4ab} \text{ for } -(a+b) < \epsilon < -(a-b),\\
\phi(\epsilon) &= \frac{1}{4ab}\int_{-b}^{+b} d\epsilon_2 = \frac{2b}{4ab} \text{ for } -(a-b) < \epsilon < +(a-b),\\
\phi(\epsilon) &= \frac{1}{4ab}\int_{\epsilon-a}^{+b} d\epsilon_2 = \frac{-\epsilon+a+b}{4ab} \text{ for } +(a-b) < \epsilon < +(a+b).
\end{aligned}\right\} \quad (3)$$

Thus it appears that in this case the graph of the resultant law of error is represented by the upper base and the two sides of a trapezoid, the lower base being the axis of ϵ and the line joining the middle points of the bases being the axis of $\phi(\epsilon)$. (See the first figure in Art. 13.) The properties of (3), including the determination of the limits, are also illustrated by the adjacent trapezoid of numerals arranged to represent the case wherein $a = 0.5$ and $b = 0.3$. The vertical scale, or that for $\phi(\epsilon)$, does not, however, conform exactly to that for ϵ.

11011
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Prob. 20. Prove that the values of $\phi(\epsilon)$ as given by equation (3) satisfy the condition specified in equation (3), Art. 10.

Prob. 21. Examine equations (3) for the case wherein $a = b$ and $b = 0$; and interpret for the latter case the first and last of (3).

Prob. 22. Find from (3), and (1) of Art. 11, the probable error of the sum of two tabular logarithms.

ART. 13. ERRORS OF INTERPOLATED VALUES.

Case I.—One of the most instructive cases to which formulas (3) of Art. 12 are applicable is that of interpolated logarithms, trigonometric functions, etc., dependent on first differences. Thus, suppose that v_1 and v_2 are two tabular logarithms, and that it is required to get a value v lying t tenths of the interval from v_1 towards v_2. Evidently

$$v = v_1 + (v_2 - v_1)\,t = (1 - t)v_1 + tv_2\,;$$

and hence if e, e_1, e_2 denote the actual errors of v, v_1, v_2, respectively,

$$e = (1 - t)e_1 + te_2. \qquad (1)$$

It is to be carefully noted here that e as given by (1) requires the retention in v of at least one decimal place beyond the last tabular place. For example, let $v = \log(24373)$ from a 5-place table. Then $v_1 = 4.38686$, $v_2 = 4.38703$, $v_2 - v_1 = +0.00017$, $t = 0.3$, and $v = 4.38691.1$. Likewise, as found from a 7-place table, $e_1 = -0.45$, $e_2 = +0.37$ in units of the fifth place; and hence by (1) $e = -0.20$. That is, the actual error of $v = 4.38691.1$ is $= 0.20$, and this is verified by reference to a 7-place table.

The reader is also cautioned against mistaking the species of interpolated values here considered for the species commonly used by computers, namely, that in which the interpolated value is rounded to the nearest unit of the last tabular place. The latter species is discussed under Case II below.

Confining attention now to the class of errors specified by equation (1), there result in the notation of the preceding article

$$\epsilon_1 = (1 - t)e_1, \quad \epsilon_2 = te_2, \quad \text{and} \quad \epsilon = e = \epsilon_1 + \epsilon_2\,;$$

and since e_1 and e_2 each vary continuously between the limits

± 0.5 of a unit of the last tabular place, a and b in equations (3) of that article have the values

$$a = 0.5(1 - t), \quad b = 0.5t.$$

Hence the law of error of the interpolated values is expressed as follows:

$$\left.\begin{aligned}\phi(\epsilon) &= \frac{0.5 + \epsilon}{(1 - t)t} \text{ for values of } \epsilon \text{ betw. } -0.5 \text{ and } -(0.5 - t),\\ &= \frac{1}{1 - t} \text{ for values of } \epsilon \text{ betw. } -(0.5 - t) \text{ and } +(0.5 - t),\\ &= \frac{0.5 - \epsilon}{(1 - t)t} \text{ for values of } \epsilon \text{ betw. } +(0.5 - t) \text{ and } +0.5.\end{aligned}\right\} \quad (2)$$

The graph of $\phi(\epsilon)$ for $t = 1/3$ is shown by the trapezoid AB, BC, CD in the figure on page 500. Evidently the equations (2) are in general represented by a trapezoid, which degenerates to an isosceles triangle when $t = 1/2$.

The probable, mean, and average errors of an interpolated value of the kind in question are readily found from (2), and from equations (1) of Art. 11, to be

$$\left.\begin{aligned}\epsilon_p &= (1/4)(1 - t) && \text{for } \quad 0 < t < 1/3,\\ &= 1/2 - (1/2)\sqrt{2t(1 - t)} && \text{for } 1/3 < t < 2/3,\\ &= 1/4t && \text{for } 2/3 < t < 1.\\ \epsilon_m &= \left\{\frac{1 - (1 - 2t)^4}{96(1 - t)t}\right\}^{1/2}. && \\ \epsilon_a &= \frac{1 - (1 - 2t)^3}{24(1 - t)t} && \text{for } \quad 0 < t < 1/2,\\ &= \frac{1 - (2t - 1)^3}{24(1 - t)t} && \text{for } 1/2 < t < 1.\end{aligned}\right\} \quad (3)$$

It is thus seen that the probable error of the interpolated value here considered decreases from 0.25 to 0.15 of a unit of the last tabular place as t increases from 0 to 0.5. Hence such values are more precise than tabular values; and the computer who desires to secure the highest attainable precision with a given table of logarithms should retain one additional figure beyond the last tabular place in interpolated values.

Case II.—Recurring to the equation $v = v_1 + t(v_2 - v_1)$ for an interpolated value v in terms of two consecutive tabular values v_1 and v_2, it will be observed that if the quantity $t(v_2 - v_1)$ is rounded to the nearest unit of the last tabular place, a new error is introduced. For example, if $v_1 = \log 1633 = 3.21299$, and $v_2 = \log 1634 = 3.21325$ from a 5-place table, $v_2 - v_1 = +26$ units of the last tabular place; and if $t = 1/3$, $t(v_2 - v_1) = 8\frac{2}{3}$; so that by the method of interpolation in question there results $v = 3.21299 + 9 = 3.21308$. Now the actual errors of v_1 and v_2 are, as found from a 7-place table, -0.38 and $+0.21$ in units of the fifth place. Hence the actual error of v is by equation (1), $\frac{2}{3} \times -0.38 + \frac{1}{3} \times +0.21 - \frac{1}{3} = -0.52$, as is shown directly by a 7-place table.

It appears, then, that in this case the error-equation corresponding to (1) is

$$e = (1 - t)e_1 + te_2 + e_3, \tag{4}$$

wherein e_1 and e_2 are the same as in (1) and e_3 is the actual error that comes from rounding $t(v_2 - v_1)$ to the nearest unit of the last tabular place.

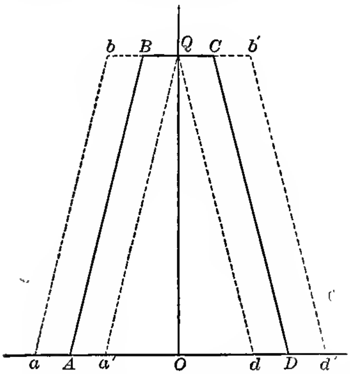
The error e_3, however, differs radically in kind from e_1 and e_2. The two latter are continuous, that is, they may each have any value, between the limits -0.5 and $+0.5$; while e_3 is discontinuous, being limited to a finite number of values dependent on the interpolating factor t. Thus, for $t = 1/2$ the only possible values of e_3 are $0 + 1/2$, and $-1/2$; likewise for $t = 1/3$, the only possible values of e_3 are 0, $+1/3$, and $-1/3$. It is also clear that the maximum value of e, which is constant and equal to $1/2$ for (1), is variable for (4) in a manner dependent on t. For example, in (4),

The maximum of	$e = 1/2 + 1/2 = 1$,	for	$t = 1/2$,
" "	" $e = 1/2 + 1/3 = 5/6$,	"	$t = 1/3$,
" "	" $e = 1/2 + 1/2 = 1$,	"	$t = 1/4$,
" "	" $e = 1/2 + 2/5 = 9/10$	"	$t = 1/5$.

The determination of the law of error for this case presents some novelty, since it is essential to combine the continuous errors $(1 - t)e_1$ and te_2 with the discontinuous error e_3. The simplest mode of attacking the problem seems to be the following quasi-geometrical one. In the notation of Arts. 12 and 13, put in (4) $e = \epsilon$, $(1 - t)e_1 = \epsilon_1$, $te_2 = \epsilon_2$, and $e_3 = \epsilon_3$. Then

$$\epsilon = (\epsilon_1 + \epsilon_2) + \epsilon_3. \quad (5)$$

The law of error for $(\epsilon_1 + \epsilon_2)$ is given by equation (2) for any value of t. Hence for a given value of t there will be as many expressions of $\phi(\epsilon)$ as there are different values of ϵ_3. The graphs of $\phi(\epsilon)$ will all be of the same form but will be differently placed with reference to the axis of $\phi(\epsilon)$. Thus, if $t = 1/3$ the values of ϵ_3 are $-1/3$, 0, and $+1/3$, and these are equally likely to occur. For $\epsilon_3 = 0$ the graph is given directly by (2), and is the trapezoid $ABCD$ symmetrical with respect to OQ. For $\epsilon_3 = -1/3$ the graph is $abQd$, of the same form as $ABCD$ but shifted to the left by the amount of $\epsilon_3 = -1/3$. Similarly, the graph for the case of $\epsilon_3 = +1/3$ is $a'Qb'd'$, and is produced by shifting $ABCD$ to the right by an amount equal to $+1/3$.



Now, since the three systems of errors for this case are equally likely to occur, they may be combined into one system by simple addition of the corresponding element areas of the several graphs. Inspection of the diagram shows* that the resultant law of error is expressed by

$$\left.\begin{aligned} \phi(\epsilon) &= (1/4)(5 + 6\epsilon) \quad \text{for } -5/6 < \epsilon < -1/6, \\ &= 1 \quad \text{for } -1/6 < \epsilon < +1/6, \\ &= (1/4)(5 - 6\epsilon) \quad \text{for } +1/6 < \epsilon < +5/6. \end{aligned}\right\} \quad (6)$$

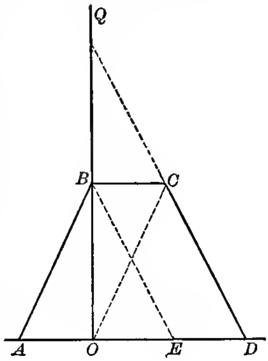
This is represented by a trapezoid whose lower base is 10/6, upper base 2/6, and altitude 1.

* Sum the three areas and divide by 3 to make resultant area = 1, as required by equation (3), Art. 10.

As a second illustration, consider equation (5) for the case $t = 1/2$. In this case ϵ_3 must be either 0 or $1/2$, the sign of which latter is arbitrary. For $\epsilon_3 = 0$, equations (2) give

$$\left.\begin{aligned}\phi(\epsilon) &= 2 + 4\epsilon \quad \text{for} \quad -1/2 < \epsilon < 0,\\ &= 2 - 4\epsilon \quad \text{for} \quad 0 < \epsilon < +1/2.\end{aligned}\right\} \qquad (7)$$

This function is represented by the isosceles triangle AQE whose altitude OQ is twice the base AE.



Similarly $\phi(\epsilon)$ for $\epsilon_3 = +1/2$ would be represented by the triangle AQE displaced to the right a distance $1/2$; and if the two systems for $\epsilon_3 = 0$ and $\epsilon_3 = +1/2$ be combined into one system, their resultant law of error is evidently

$$\left.\begin{aligned}\phi(\epsilon) &= 1 + 2\epsilon \quad \text{for} -1/2 < \epsilon < 0,\\ &= 1 \qquad\quad \text{for } 0 < \epsilon < +1/2,\\ &= 2 - 2\epsilon \quad \text{for} +1/2 < \epsilon < 1;\end{aligned}\right\} \qquad (8)$$

the graph of which is $ABCD$. On the other hand, if the errors in this combined system be considered with respect to magnitude only, the law of error is

$$\phi(\epsilon) = 2(1 - \epsilon) \quad \text{for} \quad 0 < \epsilon < 1, \qquad (9)$$

the graph of which is OQD.

The student should observe that (6), (7), (8), and (9) satisfy the condition $\int \phi(\epsilon)d\epsilon = 1$ if the integration embraces the whole range of ϵ.

The determination of the general form of $\phi(\epsilon)$ in terms of the interpolating factor t for the present case presents some difficulties, and there does not appear to be any published solution of this problem.* The results arising from one phase of the problem have been given, however, by the author in the Annals of Mathematics,† and may be here stated without proof. The phase in question is that wherein t is of the form $1/n$, n being any positive integer less than twice the greatest

* The author explained a general method of solution in a paper read at the summer meeting of the American Mathematical Society, August, 1895.

† Vol. II, pp. 54–59.

tabular difference of the table to which the formulas are applied. For this restricted form of t the possible maximum value of ϵ as given by equation (5) is, in units of the last tabular place, $(2n - 1)/n$ for n odd and 1 for n even.

The possible values of ϵ_3 of equation (5) are

$$0, \quad \pm\frac{1}{n}, \quad \pm\frac{2}{n}, \ldots \pm\frac{n-1}{2n} \qquad \text{for } n \text{ odd,}$$

$$0, \quad \pm\frac{1}{n}, \quad \pm\frac{2}{n}, \ldots \pm\frac{n-2}{2n}, \quad \pm\frac{1}{2} \qquad \text{for } n \text{ even.}$$

An important fact with regard to the error $1/2$ for n even is that its sign is arbitrary, or is not fixed by the computation as is the case with all the other errors. However, the computer's rule, which makes the rounded last figure of an interpolated value even when half a unit is to be disposed of, will, in the long-run, make this error as often plus as minus.

The laws of error which result are then as follows:

For n odd.

$$\phi(\epsilon) = 1 \qquad \text{for } \epsilon \text{ between } -1/2n \text{ and } +1/2n,$$

$$\phi(\epsilon) = \frac{n}{n-1}\left(\frac{2n-1}{2n} \pm \epsilon\right) \text{ for } \epsilon \text{ betw. } \mp 1/2n \text{ and } \mp(2n-1)/2n.$$

For n even.

$$\phi(\epsilon) = \frac{n}{2(n-1)}\left(\frac{2n-2}{n} \pm \epsilon\right) \text{ for } \epsilon \text{ between } 0 \text{ and } \mp 1/n,$$

$$= \frac{n}{n-1}\left(\frac{2n-1}{2n} \pm \epsilon\right) \text{ for } \epsilon \text{ betw. } \mp 1/n \text{ and } \mp (n-1)/n,$$

$$= \frac{n}{2(n-1)}(1 \pm \epsilon) \qquad \text{for } \epsilon \text{ between } \mp (n - 1/n) \text{ and } \mp 1.$$

By means of these formulas and (1) of Art. 11 the probable, mean, and average errors for any value of n can be readily found. The following table contains the results of such a computation for values of n ranging from 1 to 10. The maximum actual error for each value of n is also added. The verification of the tabular quantities will afford a useful exercise to the student.

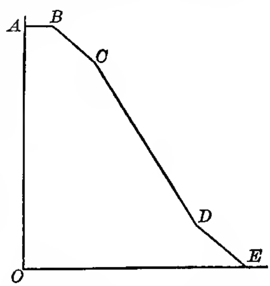
TYPICAL ERRORS OF INTERPOLATED LOGARITHMS, ETC.

Interpotating Factor. $t = 1/n$	Probable Error. ϵ_p	Mean Error. ϵ_m	Average Error. ϵ_a	Maximum Actual Error.
1	0.250	0.289	0.250	1/2
1/2	.292	.408	.333	1
1/3	.256	.347	.287	5/6
1/4	.276	.382	.313	1
1/5	.268	.370	.303	9/10
1/6	.277	.385	.315	1
1/7	.274	.380	.311	13/14
1/8	.279	.389	.318	1
1/9	.278	.386	.316	17/18
1/10	.281	.392	.320	1

When the interpolating factor t has the more general form m/n, wherein m and n are integers with no common factor, the possible values of ϵ_3 are the same as for $t = 1/n$. But equations (3) of Art. 12 are not the same for $t = m/n$ as for $t = 1/n$, and hence for the more general form of t, $\phi(\epsilon)$ assumes a new type which is somewhat more complex than that discussed above. The limits of this work render it impossible to extend the investigation to these more complex forms of $\phi(\epsilon)$. It may suffice, therefore, to give a single instance of such a function, namely, that for which $t = 2/5$. For this case

$$
\begin{aligned}
\phi(\epsilon) &= 1 && \text{for } \epsilon \text{ between } 0 \text{ and } \mp 1/10, \\
&= (5/6)(13/10 \pm \epsilon) && \text{for } \epsilon \text{ between } \mp 1/10 \text{ and } \mp 3/10, \\
&= (5/3)(4/5 \pm \epsilon) && \text{for } \epsilon \text{ between } \mp 3/10 \text{ and } \mp 7/10, \\
&= (5/6)(9/10 \pm \epsilon) && \text{for } \epsilon \text{ between } \mp 7/10 \text{ and } \mp 9/10.
\end{aligned}
$$

The graph of the right-hand half of this function is shown in the accompanying diagram, the whole graph being symmetrical with respect to OA, or the axis of $\phi(\epsilon)$.



Attention may be called to the striking resemblance of this graph to that of the law of error of least squares.

Prob. 23. Show from equations (3) that ϵ_m varies from $1/\sqrt{12} = 0.29 -$, for $t = 0$, to $1/\sqrt{24} = 0.20 +$, for $t = 0.5$; and that ϵ_a varies from 0.25 to 1/6 for the same limits.

Prob. 24. Show that the probable, mean, and average errors for the case of $t = 2/5$ cited above (p. 503) are $\pm$ 0.261, $\pm$ 0.251, and $\pm$ 0.290, respectively.

ART. 14. STATISTICAL TEST OF THEORY.

A statistical test of the theory developed in Art. 13 may be readily drawn from any considerable number of actual errors of interpolated values dependent on the same interpolating factor. The application of such a test, if carried out fully by the student, will go far also towards fixing clear notions as to the meaning of the critical errors.

Consider first the case in which an interpolated value falls midway between two consecutive values, and suppose this interpolated value retains two additional figures beyond the last tabular place. Then by equations (2), Art. 13, the law of error of this interpolated value is

$$\begin{aligned}\phi(\epsilon) &= 2 + 4\epsilon \text{ for } \epsilon \text{ between } -0.5 \text{ and } 0\\ &= 2 - 4\epsilon \text{ for } \epsilon \text{ between } 0 \text{ and } +0.5.\end{aligned}$$

Hence by equation (1) of Art. 11, or equation (3) of Art. 12, the probable error in this system of errors is $\frac{1}{2} - (\frac{1}{4})\sqrt{2} = 0.15$. It follows, therefore, that in any large number of actual errors of this system, half should be less and half greater than 0.15. Similarly, of the whole number of such errors the percentage falling between the values 0.0 and 0.2 should be

$$\int_{-0.2}^{+0.2} \phi(\epsilon)d\epsilon = 2\int_{0}^{+0.2}(2 - 4\epsilon)d\epsilon = 0.64;$$

that is, sixty-four per cent of the errors in question should be less numerically than 0.2.

To afford a more detailed comparison in this case, the actual errors of five hundred interpolated values from a 5-place table have been computed by means of a 7-place table. The arguments used were the following numbers: 20005, 20035, 20065, 20105, 20135, etc., in the same order to 36635. The actual and theoretical percentages of the whole number of errors falling between the limits 0.0 and 0.1, 0.1 and 0.2, etc., are shown in the tabular form following:

Limits of Errors.	Actual Percentage.	Theoretical Percentage.
0.0 and 0.1	33.2	36
0.1 and 0.2	30.2	28
0.2 and 0.3	19.0	20
0.3 and 0.4	13.2	12
0.4 and 0.5	4.4	4
0.0 and 0.15	51.4	50

The agreement shown here between the actual and theoretical percentages is quite close, the maximum discrepancy being 2.8 and the average 1.5 per cent.

Secondly, consider the case of interpolated mid-values of the species treated under Case II of Art. 13. The law of error for this case is given by the single equation (9) of Art. 13, namely, $\phi(\epsilon) = 2(1 - \epsilon)$, no regard being paid to the signs of the errors. The probable error is then found from

$$2\int_0^{\epsilon_p} (1 - \epsilon)d\epsilon = \tfrac{1}{2},$$

whence $\epsilon_p = 1 - \frac{1}{2}\sqrt{2} = 0.29$. Similarly, the percentage of the whole number of errors which may be expected to lie, for example, between 0.0 and 0.2 in this system is

$$2\int_0^{0.2} (1 - \epsilon)d\epsilon = 0.36.$$

Using the same five hundred interpolated values cited above, but rounding them to the nearest unit of the last tabular place and computing their actual errors by means of a 7-place table, the following comparison is afforded:

Limits of Errors.	Actual Percentage.	Theoretical Percentage.
0.0 and 0.2	35.8	36
0.2 and 0.4	27.8	28
0.4 and 0.6	18.6	20
0.6 and 0.8	12.2	12
0.8 and 1.0	5.6	4
0.0 and 0.29	49.8	50

The agreement shown here between the actual and theoretical percentages is somewhat closer than in the preceding case, the maximum discrepancy being only 1.6 and the average only 0.6 per cent.

Finally, the following data derived from one thousand actual errors may be cited. The errors of one hundred interpolated values rounded to the nearest unit of the last tabular place were computed * for each of the interpolating factors 0.1, 0.2, . . . 0.9. The averages of these several groups of actual errors are given along with the corresponding theoretical errors in the parallel columns below:

Interpolating Factor.	Actual Average Error.	Theoretical Average Error.
0.1	0.338	0.320
0.2	0.288	0.303
0.3	0.321	0.304
0.4	0.268	0.290
0.5	0.324	0.333
0.6	0.276	0.290
0.7	0.321	0.304
0.8	0.289	0.303
0.9	0.347	0.320

The average discrepancy between the actual and theoretical values shown here is 0.017. It is, perhaps, somewhat smaller than should be expected, since the computation of the actual errors to three places of decimals is hardly warranted by the assumption of dependence on first differences only.

The average of the whole number of actual errors in this case is 0.308, which agrees to the same number of decimals with the average of the theoretical errors. †

* By Prof. H. A. Howe. See Annals of Mathematics, Vol. III, p. 74. The theoretical averages were furnished to Prof. Howe by the author.

† The reader who is acquainted with the elements of the method of least squares will find it instructive to apply that method to equation (1), Art. 13, and derive the probable error of e. This is frequently done without reserve by

Prob. 25. Apply formulas (3) of Art. 12 to the case of the sum or difference of two tabular logarithms and derive the corresponding values of the probable, mean, and average errors. The graph of $\phi(\epsilon)$ is in this case an isosceles triangle whose base, or axis of ϵ, is 2, and whose altitude, or axis of $\phi(\epsilon)$, is 1.

those familiar with least squares. Thus, the probable error of e_1 or e_2 being 0.25, the probable error of e is found to be

$$0.25 \sqrt{1 - 2t + 2t^2}.$$

This varies between 0.25 for $t = 0$ and 0.18 for $t = \frac{1}{2}$; while the true value of the probable error, as shown by equations (3), Art. 13, varies from 0.25 to 0.15 for the same values of t. It is, indeed, remarkable that the method of least squares, which admits infinite values for the actual errors e_1 and e_2, should give so close an approximate formula as the above for the probable error of e.

Similarly, one accustomed to the method of least squares would be inclined to apply it to equation (4), Art. 13, to determine the probable error of e. The natural blunder in this case is to consider e_1, e_2, and e_3 independent, and e_3 like e_1 and e_2 continuous between the limits 0.0 and 0.5; and to assign a probable error of 0.25 to each. In this manner the value

$$0.25 \sqrt{2(1 - t + t^2)}$$

is derived. But this is absurd, since it gives $0.25 \sqrt{2}$ instead of 0.25 for $t = 0$. The formula fails then to give even approximate results except for values of t near 0.5.