Everymind's Simplest Representations

The Line and the Quadratic Curve

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Dedication

This text is dedicated to

Miles Bland
and
J. Pryde

for having written

Algebraical Problems
and
Exercises and Problems in Algebra

both of which have been well and truly plundered
in the creation of this text.

I have tried to take things a bit further than these men did.
But we all stand on the shoulder of giants --
even if some giants are smaller than others.

(I'm quite a small giant myself. Almost midget-like.)

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Preface

I'm not going to lie to you -- this is a book about word problems. I don't like them. Augustus De Morgan didn't like them. Isaac Todhunter didn't like them. You shouldn't either. If you do like them, keep it to yourself. De Morgan wrote:

> *The reduction into equations of such problems as are usually given in the treatises on algebra rarely occurs in mathematics. ... [N]o student need give a great deal of time to it. Above all, let no one suppose, because he finds himself unable to reduce to equations the conundrums with which such books are usually filled, that, therefore, he is not made for the study of mathematics, and should give it up.*

But mathematics, unless it is simply about itself, is also about representing states of affairs in the world. We use the form of number to model the world. And De Morgan, Todhunter, and I agree that this is both useful and worthwhile. It just shouldn't be a full-time job.

This text examines the simplest of these representations which use the form either of a line or of a quadratic curve. There isn't much to either one of these forms, three terms max. So given a paragraph of conundrum, we should be able to discern which of the puzzlie-bits go to which term without straining our brains. And, as best we can, let's come up with a methodical way to do this so that laziness can have its rightful place in our work.

Introduction

If we are ever going to model anything with the form of number, we should probably start with the easiest forms and work our way up. The two easiest forms are the ones that you find in public school algebra texts: the lime and the cocoanut. Or rather, the line and the quadratic curve. (A good writer knows how to handle his typos.) We know that these are the easiest forms to model things with because public school students are sometimes rather dim, like their teachers, and with this in mind, if there were easier forms, these textbooks would start with those easier forms instead. So think about what we've got here. A line:

$$y = mx + c$$

and a bendy line:

$$y = ax^2 + bx + c$$

Take your average Euclidean plane. Draw a line on it. Pick an arbitrary midpoint. Spin it. And wherever it stops, it's still a line. If you bend it uniformly before you spin it, you have a parabola, which is a single quadratic bendy line. And spinning it doesn't change its underlying form either.

In both cases, we can forget the y because that's what we get for messing with this stuff in the first place -- it's a freebie. And we can forget the x, including when we square it, because it's the unknown. So by definition, we can't know anything about it. It's obviously mysterious.

So in the first case, we only need to find the m and the c. The m is the slope of the line and the c is the line's intercept on the Y-axis. In the second case, we have, weirdly, a parabola with a line added to it. The ax^2 is a parabola. The bx + c is a line. And this line functions as a distortion function on the parabola. The b takes every ax^2 and makes it b times x bigger or smaller. And then the c moves the resulting point up or down. We don't usually add lines and parabolas. But it's fun to think about. We usually multiply two binomials to wind up with a quadratic. Which is clearly less fun.

The conclusion I come to is that these forms can only represent simple things which can be described by a slope and a point or two points or two binomial factors giving us some kind of straight or bendy line. There are tons of such representations in old-school math texts misrepresented as "word problems". These actually represent states of affairs that can be described using only two or three mathematical terms. So let's take a bunch of examples and organize them by how they hide their puzzlie-bits.

The upside here is that we can use what we learn to actually model simple states of affairs in the world with these two simple forms of number. Then we can go look for more interesting stuff to model using much more interesting forms.

Instructions

Learners

This is not a normal text book full of problems for you to figure out. Books like that are big and heavy and full of color pictures. Here is a color picture of a quadratic equation, they say. These pictures help no one except the text book companies. No one learns math from colored pictures. Just look at the graduates these colored pictures produce. They can hardly do any math at all. We all know this. Math is about words and symbols and ideas. Here is a color picture of my large white x. See how that works? Now you are thinking about what the variable x stands for. Thinking about ideas is good. (Good luck on solving for that white x.) Also, a color picture may be worth a thousand words. But those have to be words you already know. Can you actually get a thousand of the words you already know to fit every picture you see? And what do you learn from that? We'll skip the pictures and the normal problems.

This book is about the forms representations can take when they are based on straight or bendy lines. So for each method we can find, there will be a set of examples for you to study. Read each problem. Figure out how you are going to assign the variables. At first, you will have to guess. So guess. Then check to see if your assignment is the one used here before you continue. It's simpler that way. Then try to solve the problem. Whether you solve it or not, study the solution. The point of this text is to help you learn how to discern where the straight or bendy line lies in the details of some state of affairs in the world, how to assign the variables to your representation, how to build the right model and work out its solution. These are all three important. And the first two are way harder than the third.

This will not be a long book because this stuff isn't interesting enough to go on and on about. You and I both have better things to do. But learning how to model something with mathematics is definitely worthwhile. And to do this you only have to make a genuine effort to understand each problem, check the variable assignment, solve as far as you can, and study the solution. It's actually not about getting the right answer. If x is the weight of a fish, no one actually cares about how much this fictional fish weighs. You will never see this fish again. But you might need to model something like it some day using a line or a quadratic. And by slowly coming to an understanding of these methods, you will be able to do just that.

The most important questions are:

1. Where are the puzzlie-bits?
2. Am I correctly assigning the variables?
3. Am I algebraically constructing the correct line or quadratic?

Everything else is arithmetic.

Teachers

Please, please, please, do not teach these forms of solution as if they were ecclesiastic dogma. For instance, I have seven forms for solving a word problem that uses a line. But there are really only three forms of this: sum, ratio, and x+n-x. And that might boil down to only two forms. The point is that these forms are all heuristics for grasping what is important in the data so you can construct a simple first or second degree equation.

So it would be stupid to ask students to find such and such a form in a problem or which form a problem takes or similar. Most of the "forms" are in every problem. All problems using a line are also problems of the intersection of two lines. Everything is everything. The best approach is to acquaint them with the simplest form, give problems where that simplest form is easily applied, and build up from there. We only want to find the line, construct it, and then simply solve it. These forms are just means to an end.

I have given solutions and explanations for every problem in this text. Every algebra text in book or PDF format will be chock full of these kinds of problems. And you won't have any difficulty picking out examples to match this or that form of solution. I recommend the PDF of Todhunter's *Algebra for the Use of Colleges and Schools* for good examples of these kinds of problems. Todhunter shared De Morgan's disdain for puzzlie word-problems and only offers intelligent examples that actually increase your understanding. There is also a separate handy answer-key volume to his book. Both are available on archive.org.

The problems in this book are more or less two hundred years old. If you look at several of these problem books, you will see that many of the problems make the rounds with only slight changes. I have left the spelling, punctuation, and grammar in the 18th century. A little discussion of these archaic forms will relieve the tedium of doing too many stupid word-problems.

The Line

We are going to look at various states of affairs, which are presented as word problems, and fit them to forms of solutions. No state of affairs in the world of experience has a single, unique form of solution. These first forms are means of fitting the data into the form of a line. We try to grasp the data and its relations as simply as possible in order to do this.

Note that there is a linear form of solution called **two couriers**, where two somethings are traveling in space or time and you have to figure out when or where they meet or when one is twice as old as the other. The two couriers form can become quite complex. And it can be nuanced, in that the two somethings may have already met or may never meet. This form is handled in depth in *Everymind's De Morgan's Elements* and we won't repeat any of that here.

Line

The simplest representation using the form of a line is a **line**. You look at the problem and you say, "Hmm, line," or, "One step away from being a line." They look like this:

1. What number is that, to the double of which if 18 be added the sum will be 82?

Let x = the number required.

Then by hypothesis, $2x + 18 = 32;$
(-18) $2x = 14$
(÷2) $x = 7$

Any line is $y = mx + c$. Here our x is doubled, or has an m of 2, and the c is clearly 18.

2. What number is that, to the double of which if 44 be added, the sum is equal to four times the required number?

Let x = the number

Then by hypothesis, $2x + 44 = 4x$
(- 2x) $44 = 2x$
(÷2) $22 = x$

Same representation as #1. It is also the **balance** form below. But here it's so simple, it's just a **line**.

3. What number is that, the double of which exceeds its half by 6?

Let x = the number

Then by hypothesis,	$2x - x/2 = 6$
(×2)	$4x - x = 12$
(-)	$3x = 12$
(÷3)	$x = 4$

If the form of the problem does not simply appear in your mind, write out the hypothesis. Having written it, you can see that when you combine the x's you have an m of (2 - 1/2) and a c of (-6). This line takes the form

$$1.5x - 6 = 0$$

which means that the solution is the X-intercept of the line.

ΣX

This form is pronounced "sum x" and, in these problems, as soon as you identify the unknown, you see that there are two or more groups which must be combined to make the unknown. You need to "sum" all these "x" together to make the data fit the form of a line. The line gives you the x, which often has to be put back into its summed bits to give you the required values. Note that the coefficient of the sum is again the slope of your model's line. You'll see what I mean.

1. A gentleman meeting 4 poor persons distributed five shillings amongst them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

First, you have to know that 12d. = 1s. where d are pence and s are shillings. So you might as well make the 5s. into 60d. to begin with and avoid a fractional mess of s.

Let x, 2x, 3x, 4x = pence given to 1st, 2d, 3d, 4th poor person

(laying out the x) $x + 2x + 3x + 4x = 60$
(summing the x) $10x = 60$
(÷10) $x = 6$

∴ our four poor fellow humans got 6d., 12d., 18d., 24d. each.

2. A Gentleman dying bequeathed a legacy of **L**.140 to three servants. A was to have twice as much as B; and B three times as much as C. What were their respective shares?

Here, **L.** is the pound sign for the old pound sterling and s. is shillings.
There are 20s. in **L**.1.

Let x = C's share

∴ 3x = B's share
and 6x = A's share

(laying out and summing) $(6x + 3x + x) = 10x = 140$
(÷10) $x = 14$

∴ A, B, C received **L**. 84, **L**. 42, **L**. 14

3. Four Merchants entered into a speculation, for which they subscribed **L**.4755; of which B paid three times as much as A; C paid as much as A and B; and D paid as much as C and B. What did each pay?

Let x = number of pounds A paid

∴ 3x = number B paid
4x = number C paid
7x = number D paid

(laying out and summing) $(x + 3x + 4x + 7x) = 15x = 4755$
(÷ 15) $x = 317$

∴ they contributed 317, 951, 1268, and 2219 pounds respectively.

For figuring out how many x's go with each Merchant or similar, it is faster to calculate if you write each bit down than if you try to hold it all in your head to get the multipliers. A is x. So B is 3x. So C is (A and B) is 4x. And D is (C and B) is 7x. And you see it as you write them down.

4. A Draper bought three pieces of cloth, which together measure 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each?

Let x = number of yards in first piece

∴ $x = 15 = 2d$ piece
∴ $x + 15 + 24 = x + 39 = 3d$ piece
∴ $x + x + 15 + x + 39 = 159$
(algebrate) $3x = 105$
(÷3) $x = 35$

∴ lengths 35, 50, 74

A purer taxonomist would come up with another form than ∑x to put this in. But we don't need a bunch of forms. We need a working heuristic. If you have several piles of x's, you don't care if they are multiples of each other (x, 2x, 5x) or just different (x, x+5, x+23). We just find the x's and sum them.

5. A cask which held 146 gallons, was filled with a mixture of brandy, wine, and water. In it were 15 gallons of wine more that there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Let x = gallons of brandy

∴ x + 15 = gallons wine
∴ 2x + 15 = gallons water
∴ x + x + 15 + 2x + 15 = 146
(algebrate) 4x = 116
(÷4) x = 29

∴ 29 gallons of brandy, 44 of brandy, 73 of water

6. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

You should have noticed by now that in this form of representation we start by using the simplest of the summed parts as x.

Let x = sum received by fourth man

∴ x + 4 = sum of 3d man
∴ x + 7 = sum of 2d man
∴ x + 9 = sum of 1st man.

∴x + x + 4 + x + 7 + x + 9 = 32
(algebrate) 4x = 12
(÷4) x = 3

∴ they received 12, 10, 7, and 3 shillings respectively.

7. A Father taking his 4 sons to school, divided a certain sum amongst them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Let x = share of youngest

∴ x + 9 share of 3d
∴ x + 21 share of 2d
∴ x + 39 share of eldest

∴ x + x + 9 + x + 21 + x + 39 = 7x + 6
(algebrate) 63 = 3x
(÷3) 21 = x

∴ each received 21s., 30s., 42s., and 60s. respectively.

Now I could move this problem down into the **balance** section. But no state of affairs in the world is strictly a **∑x** or a **balance** or anything else. We grasp it as fitting into one of our forms of representation. And almost every time we pick a form, we have to fudge something in the data to make it fit. So here we didn't have a total on the RHS. But we can use the 7x + 6 as the total.

Note that if we simplify it to 4x + 69 = 7x + 6, we are asking where two lines intersect, where lines have the form y = ax + b. So they meet at point (21,153) as you will see from putting 21 into either side of this last eqn.

8. A sum of money was to be divided amongst six poor persons; the second received 10d. the 3d 14d. the fourth 25d. the fifth 28d. and the sixth 33d. less than the first. Now the sum distributed was 10d. more than the treble of what the first received. What did each receive?

Let x = sum of 1st person

∴ x - 10 = sum of 2d
∴ x - 14 = sum of 3d
∴ x - 25 = sum of 4th
∴ x - 28 = sum of 5th
∴ x - 33 = sum of 6th

∴ 6x - 110 = 3x + 10
(algebrate) 3x = 120
(÷3) x = 40

∴ They received 40, 30 26, 15, 12, 7 pence respectively.

8. It is required to divide the number 99 into 5 such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16.

Let x = 1st part

∴ x - 3 = 2d
x + 10 = 3d
x - 9 = 4th
x + 16 = 5th

∴ x + x - 3 + x + 10 + x - 9 + x + 16 = 99
(combine) 4x + 14 = 99
(- 14) 5x = 85
(÷5) x = 17

∴ the parts are 17, 14, 27, 8, and 33

9. What two numbers are those whose sum is 59, and difference 17?

Let x = the smaller number

∴ x + 17 = greater number
∴ x + x + 17 = 59
2x = 42
x = 21 = smaller number

∴ 38 = greater number

Here your two different piles of x are determined by their difference.

10. At a certain election 943 men voted (women couldn't vote yet back then), and the candidate chosen had a majority of 65. How many voted for each?

Let x = loser's votes

∴ x + 65 = winner's votes
∴ x + x + 65 = 943
2x = 878
x = 439

∴ the numbers of votes are 439, 504

11. A Mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together were 142 yards. What was the length of each piece?

First note that all four pieces were equal.

Let x = the length

∴ x - 19 = length of all three of the first remnants
∴ x - 17 = length of last piece

∴ 3(x - 19) + (x - 17) = 142 = all the remnants
3x - 57 + x - 17 = 142
4x = 216

∴ x = 54 = original length of each of the four equal pieces

12. Bought 12 yards of cloth for **L**.10.14s. For part of it I gave 19s. a yard and for the rest, 17s. a yard. How many yards of each were bought?

L.10.14s in shillings is 10·20 + 14 = 214s. Always do this to simplify things, avoiding fractions of, in this case, pounds sterling.

Let x = number at 19s. a yard

∴ 12 - x = number at 17s.
∴ 19x = price of the former
∴ 17(12 - x) = price of the latter

∴ 19x + 204 - 17x = 214
2x = 10
x = 5

∴ 5 yards at 19s and 12 - 5 = 7 yards at 17s.

13. A and B began to trade with stocks. In the first year A tripled his stock, and had **L**.27 to spare; B doubled his stock, and had **L**.153 to spare. Now the amount of both their gains was five times the stock of either. What was this stock?

Let x = the stock

Then 3x + 27 = A's stock after one year
$\therefore$ 2x + 27 is gain above his original x
and 2x + 153 = B's stock after one year
$\therefore$ x + 153 = B's gain

$\therefore$ 5x = 2x + 27 + x + 153 [1]
2x = 180
x = 90

Note that [1] **says** five times the stock of either = A's gain + B's gain.
Every algebraical statement up until the steps of computation **mean something**.

Also, you could call this a balance form or some other form. But in the doing of it, you are simply looking for some stuff to sum up and set equal to something. And what that stuff is cannot be seen until you start putting the statements together as little bits of algebra.

14. A Countryman had two flocks of sheep, the smaller consisting entirely of ewes, each of which brought him 2 lambs. On counting them he found that the number of lambs was equal to the difference between the two flocks. If all his sheep had been ewes, and brought forth 3 lambs apiece, his stock would have been 432. Required the number in each flock.

Let us note that, the reality of stock-breeding being what it is, that if his stock had all been ewes, there would have been no rams, and therefore no lambs. But let us proceed.

Let x = number in lesser flock

$\therefore$ number of lambs = 2x = difference between flocks
$\therefore$ 3x = number in greater flock

$\therefore$ 4x + (3·4x) = 432
16x = 432
x = 27

$\therefore$ flocks have 27 and 81 sheep

15. A General, whose horse was 1/3 of his foot, after a defeat found, that before the battle 1/12 - 120 of his foot and 1/12 + 120 of his horse had deserted; 1/4 of his whole army was in garrison; and 3/8 remained, the rest being taken prisoners or slain. Now 300 + the number slain = 1/2 of the foot he had at first. Of how many of each and total was his army?

Let x = number of horse

∴ 3x = number of foot
∴ 4x = total
∴ 3x/2 - 300 = number slain

∴ x/4 - 120 + x/12 + 120 + x + 3x/2 + 3x/2 - 300 = 4x
What are the 5 elements of the LHS of that last line?
3x + x + 48x - 3600 = 48x
4x = 3600
x = 900

∴ 900 cavalry, 2700 infantry for a total of 3600 men

16. Two persons A and B both have the same annual income. A lays by one-fifth of his; but by B spending **L**.80 per annum more than A, at the end of 4 years finds himself **L**.220 in debt. What did each receive and spend annually?

Let 5x = their annual income
Why 5x?

∴4x = A's annual expenditure
4x + 80 = B's annual expenditure

∴ 4x + 80 - 5x = 80 - x = B's annual debt
∴ 320 - 4x = 220
4x = 100
x = 25

∴ their annual income is 125 pounds
A's annual expenditure is 100 pounds and B's is 180 pounds

Redux

The problems so far have been Bland's. Let's go back to simpler problems using Pryde's book and work our way back up. This time I will state the problem and assign the variable. Then I will give a few hints and, finally, the answers. Your task is to use the data and the hints to construct the line or quadratic equation which produces the given answer. I will give fewer hints as we go along. The only point to all these exercises is learning how to create a representation of the data using a line or a quadratic. The rest is arithmetic and you can already do that.

17. A gentleman bequeathed **L**.210 to two servants; to one he left half as much as to the other; what did he leave to each?

2x = what the first received and you don't need a hint.

Answer: 70, 140

18. A prize of **L**.864 was divided between two persons, A and B, whose shares therein were 5 to 7; what was the share of each?

5x = A's share which pretty much tell's you what B's is.

Answer: A's share is 360 and B's 504

19. A sum of money is to be shared between A and B, in such a manner that as often as A gets 10 pounds, B gets 7; now A received **L**.30 more than B; find the sum shared, and the share of each.

10x = A's share and set them equal handling the 30 correctly.

Answer: A's share was 100, B's 70. Total 170.

20. A bankrupt owed two creditors **L**.560: the difference of the debts was to the less as 4 to 5; what are the debts?

4x = the difference
If A > B and A - B = C, we have C : B :: 4 : 5

Answer: **L**.200 and **L**.360

21. Divide the number 198 into five such parts, that the first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four and the fifth divided by five, may all be equal.

Let x be the common element in the five parts. Then the first part will be x - 1 from

1st + 1 = 2d + 2
1st = 2d + 1
1st - 1 = 2d

Figure out why the 2d part is x - 2 and you should be able to just read off the other parts, and sum them as equal to 198.

Answer: 23, 22, 27, 6, 120

Balance

The balance form comes from having two equal lines in the same state of affairs. As with ∑x, you have more than one instance of the same unknowns but with their details correctly applied they are equal to each other like weights in a balance. Or think of it as having a bunch of x-piles which naturally fall into two piles instead of one. Then you set them equal to each other.

22. A Bookseller sold 10 books at a certain price; and afterwards 15 more at the same rate. Now at the latter time he received 35 shillings more than at the former. What did he receive for each book?

Let x = the price of a book

Then 10x = the price of the first set
and 15x = the price of the second set

By hypothesis,	$15x = 10x + 35$
(- 10x)	$5x = 35$
(÷5)	$x = 7$

So 7s. per book.

23. What number is that, the treble of which increased by 12, shall as much exceed 54 as that treble is below 144?

Let x = the number

∴ $3x + 12 - 54 = 144 - 3x$

(algebrate)	$6x = 186$
(÷6)	$x = 31$

By the time these states of affairs get this complicated, you need a diagram. And in algebra, a diagram is an algebraic statement. Number trebled and increased by 12 is (3x + 12). The excess of a number m over 54 is (m - 54). So m is the trebled thing ∴ (3x + 12 - 54) gives you your LHS.

24. Two persons begin to play with equal sums of money; the first lost 14 shillings, the other won 24 shillings and then the second had twice as many shillings as the first. What sum had each at first?

Let x = the sum

Then after playing they had x - 14 and x + 24
Which gives us x - 14 = x + 24 for 1st guy = 2d guy

We have set things up here. But it's not really equal.
Because 2d guy ended up with twice the total of the first

∴ 2(x - 14) = x + 24
(distribute) 2x - 28 = x + 24
(algebrate) x = 52

Make sure you understand how 2d guy's having twice the 1st guy's money caused us to multiply the 1st guy's money by two to get equality. It is easy to get that backwards.

25. Two Robbers after plundering a house found that they had 35 guineas between them; and that if one of them had had 4 guineas more, he should have had twice as many as the other. How many had each?

Let x be the number one had.

∴ 35 -x is the number the other had
∴ 35 - x + 4 = 2x
∴ 39 = 3x
∴ 13 = x

∴ They had 13 and 22 guineas respectively.

You couldn't solve this without playing around with the data a bit. Do not try to model the world in your head. Do it on paper. Once you have the 35 - x, you can use the 4. And only then do you have the 2x to balance it with.

Mathematics is the **science of diagrams**. And little algebraic statements are the elements of an algebraic diagram. In a sense, this is all you have to work with. And only as the bits appear on paper can you think about their relations.

26. A Farmer has two flocks of sheep, each containing the same number. From one of these he sells 39 and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

Let x = number required

∴ x - 39 and x - 93 are what remain and the latter is smaller
∴ x - 39 = 2(x - 93)
x - 39 = 2x - 186
x = 147

∴ Each flock had 147 sheep.

27. After A had lost 10 guineas to B, he wanted only 8 guineas in order to have as much money as B; and together they had 60 guineas. What money had each at first?

Let x = A's original sum

∴ 60 -x = B's original sum

Then after playing A has x - 10 and B has 70 - x

∴ x - 10 + 8 = 70 - x
2x = 72
x = 36

∴ They had 36 and 24 guineas to begin with.8

28. Two workmen A and B were employed together for 50 days, at 5s. per day each. A spent 6d. a day less than B did, and at the end of the 50 days he found he had saved twice as much as B, and the expense of two days over. What did each spend per day?

Let x = what A spent per day in pence (5s. = 12d.)

∴ 60d. (12d./s. · 5) = what A saved per day
∴ 54 - x = what B saved per day

∴ 3000 - 50x = 5400 - 100x + 2x   What does each term in this line **mean**?
48x = 2400
x = 50

∴ A spent 50d./day and B spent 56d./day

29. A and B began to trade with equal sums of money. In the first year A gained 40 pounds and B lost 40; but in the second A lost ⅓ of what he then had, and B gained a sum less by 40 pounds than twice the sum A had lost; when it appeared B had twice as much money as A. What money did each begin with?

Let x = the beginning equal sum

∴ x + 40 = A's sum after 1st year
x - 40 = B's sum after 1st year
2/3·(x + 40) = A's total after 2d year
(x - 40) + 2/3·(x + 40) - 40 = B's 2d year total

∴ 2·(2/3·(x + 40)) = (x - 40) + 2/3·(x + 40) - 40 What do the terms here **mean**?
2/3(x + 40) = x - 80
2x + 80 = 3x - 240
320 = x

∴ Both began with 320 pounds.

30. A and B being at play severally cut packs of cards so as to take off more than they left. Now it happened that A cut off twice as many as B left, and b cut off seven times as many as A left. How were the cards cut by each?

Let 2x = number A cut

Then 52 - 2x = number A left
∴ x = number B left
∴ 52 - x = number B cut off

∴ 52 - x = 7(52 - 2x)
52 - x = 364 - 14x
13x = 312
x = 24

∴ A cut 48 and B cut 28

The simplest value in this problem is x. Above, this was taken as what B left. But A could take x leaving 52 - x. Then B must take 7(52 - x) and then

x = 2(52 - (7(52 - x)))

and that will work out as x = 48 for A and then B = 7(52 - 48) = 28.

Here is the important thing: Everything you write down must **mean something** with respect to the state of affairs you are representing. You must be able to say each algebraic statement in **words relating to the problem** being modeled.

31. What number is that whose one-third part exceeds its one-fourth part by 16?

We will later have a form of **fraction x** where we would use ¼ and ⅓ in this problem. But we can use least common multiples for the same thing: lcm(3,4) = 12

Let 12x = the number

∴ $4x - 3x = 16$
$x = 16$

∴ The number is $12 \cdot 16 = 192$.

Using LCM instead of fractions simplifies the arithmetic, if nothing else.

32. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew 4 gallons, and then found there were four times as many gallons in the larger as in the smaller. How many gallons were there in each at first?

Let the casks contain x and 3x gallons

∴ $4(x - 4) = 3x - 4$
$x = 12$

∴ Casks held 36 and 12 gallons.

33. A sum of money was divided between A and B, so that the share of A was to the share of B as 5 to 3 and A's share exceeded 5/9 of the whole by 50 pounds. What was the share of each person?

This is an example of data which has a ratio (5:3 or 5/3) but does not really fit the ratio form of solution.

Let 5x = A's share

∴ 3x = B's share
∴ 8x = whole sum

∴ $5x = 5/9 \cdot 8x + 50$
$45x = 40x + 450$
$x = 90$

∴ Shares were 540 and 270.

Note that you needed the 3x to get the total of 8x but did not need 3x in the equation.

34. Being sent to market to buy a certain amount of meat, I found that if I bought beef, which was 4d. a pound, I should lay out all the money I was entrusted with; but if I bought mutton which was three pence halfpenny a pound, I should have 2s. left. How much meat was sent for?

We're going to use 2x for the number of pounds. Try to figure out why before you go on.

Let $2x$ = number of pounds

∴ $8x$ = price of $2x$ pounds of beef
$7x$ = price of $2x$ pounds of mutton

∴ $8x = 7x + 24$ where 24d. = 2s.
$x = 24$

∴ 48 lbs. of meat sent for

If you double the mutton price of 3½d. you get 7. So double the beef to 8. So double the x to 2x. And Bob's your uncle. Just remember you are solving for 2x in the end, not for x.

You could also set this up as $4x = 7/2 \cdot x + 24$. His way clears the fractions before you start. This way you'd have to clear them on paper. But this way might be simpler.

35. A Fish was caught, whose tail weighed 9lbs.; his head weighed as much as his tail and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x$ = number of lbs. the body weighed

∴ $9 + x$ = weight of tail

∴ tail + head = $9 + 9 + x = 2x$ = body
∴ $x = 18$

∴ 9 (tail) + 27 (head) + 36 (body) = 72lbs.

Here, 2x was chosen because the data said "and half his body." It is a simple way to dump fractions.

36. Some persons agreed to give sixpence each to a waterman for carrying them from London to Gravesend; but with this condition, that for every other person taken in by the way, three pence should be abated from their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but five pence each. How many persons were there at first?

Let 4x = number of passengers at first   Why 4x?

∴ x + 3 more taken in
∴ 3x + 9 = sum deducted from joint fare

∴ 24x - (3x + 9) = 20x
x = 9

∴ 36 original passengers

Can you say what the 24x and the 20x mean?

37. In a mixture of wine and cyder, half of the whole plus 25 gallons was wine, and one third of the whole minus 5 gallons was cyder. How many gallons were there of each?

You want the whole to be LCM from ½ and ⅓.

Let 6x = number of gallons in all

∴ 3x + 25 = gallons of wine
2x - 5 = gallons of cyder

∴ 6x = 3x + 25 + 2x - 5
x = 20

∴ 85 gallons of wine and 35 of cyder

38. A and B engaged in trade, A with **L.**240 and B with **L.**96. A lost twice as much as B; and upon settling their accounts it appeared that A had threes times as much remaining as B. How much did each lose?

Let x = B's loss

∴ 96 - x = what B still has
∴ 2x = A's loss
∴ 240 - 2x = what A still has

∴ 240 - 2x = 3(96 - x)
x = 48

∴ A lost **L.**96 and B lost **L.**48

39. A General having lost a battle found that he had only half his army plus 3600 men left, fit for action; one-eighth of his men plus 600 were wounded, and the rest or one-fifth of the army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Let x = total of army

∴ x/2 + 3600 = fit for service
x/8 + 600 = wounded
x/5 = remainder

∴ x = x/2 + 3600 + x/8 + 600 + x/5
3x/8 - x/5 = 4200
15x - 8x = 7x = 168000

∴ x = 24000

40. A, B, and C enter into partnership; A paid in as much as B plus one-third of C; B paid in as much as C and one-third of A; and C paid in **L**.10 and one-third of A. What did each man contribute to the stock?

Let 3x = A's contribution

∴ 10 + x = C's contribution
10 + 2x = B's contribution

∴ $3x = 10 + 2x + (10 + x)/3 = B + \frac{1}{3}C$
$2x/3 = 10 + 10/3$
$2x = 40$
$x = 20$

∴ A,B,C contributed 60, 50, and 30 pounds

41. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value: but being detected, the pieces were found to be really worth no more than 8 guineas. What was their original value?

1 guinea = 21s.
half a crown = 30d. = 5 sixpences

Calculate value in sixpences.

Let x = number of 6d. each coin was worth

∴ x - 5 = number of 6d. each worth after filing

∴ $16(x - 5) = 16 \cdot 21 = 416$
$16x = 416$

x = 26 sixpence each or 13s.

42. A Gentleman gave in charity **L**.46; a part thereof in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now a man and a woman had between them **L**.8. What was given to the men and what to the women?

Let 5x = pounds men received

∴ 46 - 5x = pounds women received
x = one man's money
8 - x = one woman's money

∴ 56 - 7x = 46 - 5x What does the LHS **mean**?
2x = 10
x = 5

∴ men received 25 pounds and women 21 pounds

You could also set this up as

$$1/5 \cdot x + 1/7(46 - x) = 8$$

and it would give you the total for men for x. Or you could do

$$46 - 5x = 7(8 - x)$$

which I will let you interpret for yourself.

43. A Landlord let his farm for **L**.10 a year in money and corn-rent. When corn sold at 10s. a bushel, he received 10s. an acre from his land; but when it sold for 13s. 6d. a bushel, he received 13s. an acre. Of how many bushels did the corn-rent consist?

Let x = number of bushels

first case:
∴ 10x + 200 = annual income in shillings (10 pounds = 200s.)
∴ x + 20 = number of acres

second case:
13.5x + 200 or 27/2 · x + 200 then divide by 13 and rationalize
(27x + 400)/26 = number of acres

∴ (27x + 400)/26 = x + 20
27x + 400 = 26x + 520

∴ x = 120 = number of acres

44. When the price of a bushel of barley wanted but 3d. to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for four bushels of barley and 7s. 6d. in money. What was the price of a bushel of each?

Work from the ratio to assign the variable. Move everything to pence.

5x = price of bushel of oats

∴ 8x - 3 = price bushel of barley

∴ 45x = 32x - 12 + 90
13x = 78
x = 6

∴ prices are 30d. for oats and 45d. for barley

You could also ge the price of barley this way:

x = barley price
x + 3 : 4/9·x + 10 :: 8 : 5
∴ 5(x + 3) = 8(4/9·x + 10) and algebrate.

You will see more proportions like this later.

45. A Gentleman had some of his horses at grass at 3s. a week, and the rest at livery stables at 10s. a week. The horses in the stables cost him twice as much a week as the horses at grass. But he finds that if he had sent 3 horses to grass out of the stables, the expense of the stables would have been only 6s. a week more than the grass. How many horses had he?

Let x = horses at grass

∴ 3x = weekly expense grass
6x = weekly expense stables
∴ 6x/10 = number horses in stable
6x - 30 = expense of stables after 3 horses sent to grass
3x + 9 = expenses of grass for the switched horses

∴ 6x - 30 = 3x + 9 + 6 = 3x + 15
3x = 45
x = 15

∴ 15 horses at grass and 9 in stables

46. A Silversmith received in payment for a certain weight of wrought plate, the price of which was **L**.10, the same weight of unwrought plate and **L**.3. 15s besides. At another time he exchanged 12oz. of wrought plate of the same workmanship as before for 8oz. of unwrought (for which he allowed the the same price as before), and **L**.2. 16s. in money. What was the price of wrought plate per ounce, and the weight of the first sold?

Let x = number of ounces

∴200/x = price of wrought oz.
125/x = price of unwrought oz.

∴ 2400/x = 1000/x + 56
1400/x = 56
25/x = 1

∴ x = 25 = ounces and the price was 8s. per ounce

47. In changing a bill of **L**.85 into guineas (1g. = 21s.) and shillings (the number of shillings being ¼ number of guineas) on examination they all proved adulterated below the standard value, to the amount in the whole of **L**.8. 5s. To make up the deficiency, nine more such guineas were paid; and four such shillings and three good ones returned. Required the number and value of the guineas and shillings paid at first.

Give it a try and when your brain hurts come see one way to do it.

L.85 = 85s.·20 = number s. + 4·21·number s. = 85·number

∴ 20 = number of shillings
80 = number of guineas

Make sure you understand everything so far.
Let x = value adulterated guineas

∴ (1700 - 165) = 1535 = 80x + 20·value of adulterated shilling

∴value adulterated shilling = (1535 - 80x)/20 = (307 - 16x)/4

∴ 165 = 9x - 3 - 307 + 16x = 25x - 310
25x = 475
x = 19

∴ bad guinea worth 19s. ∴ bad shilling = (307 - 16·19)/4 = 3/4 = 9d.

You will gain more goodness than you would expect by working with that last problem until you understand the **meaning** of every line.

48. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every 3 guns and 10 over. Also, the whole number of hands was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were one-fourth of the survivors, there wanted 5, to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Let $3x$ = number of guns

$\therefore 22x + 10$ = number of sailors
Because sailors + soldiers = $5\cdot$soldiers + $15x$
$\therefore$ soldiers = $\frac{1}{4}(22x + 10 - 15x) = (7x + 10)/4$
$\therefore$ all men = $(7x + 10)/4 + 22x + 10 = (95x + 50)/4$
$\therefore$ survivors = $4/3(\,(95x + 50)/4\,) = 19x + 10$
$\therefore 3x/2(13 - 5) = 19x + 10$

$\therefore x = 30$
$\therefore$ guns = $3x = 90$
seamen = $30\cdot 22 + 10 = 670$
soldiers = $(7\cdot 30 + 10)/4 = 55$

It is insane what one can do with these kinds of problems if you become good at it. This next one is the kind that would appear in the Cambridge Tripos which you would take after studying maths. See what you can do.

49. Water flows uniformly into a cistern, capable of containing 720 gallons, through a pipe; and at the same time is discharged by a pump, worked by three men, who take four strokes in a minute; but this not being sufficient, the cistern becomes full in 6 hours; they therefore now put in another pump, of such power that the quantity discharged at one stroke by this pump is to the quantity discharged at one stroke by the former :: 2 : 3; but being obliged to detach one of their number to work the pump, the former group makes only 10 strokes in 3 minutes, and the latter 5 strokes in 2 minutes; by which means the cistern is emptied in 12 hours. How much water was discharged by each pump at one stroke? and how much flowed in through the pipe in one minute?

Let $3x$ = number of gallons discharged by 1st pump at one stroke

$\therefore 2x$ = number discharged by 2d pump
$12x$ = quantity discharged by 1st in one minute with 3 men
$\therefore 6 \cdot 60 \cdot 12x$ = quantity discharged in 6 hours
$\therefore 6 \cdot 60 \cdot 12x + 720 = 720(6x + 1)$ = quantity introduced by pipe in that time

Add additional pump:
$10x$ = discharge 1st pump in one minute
$5x$ = discharge 2d pump in one minute
$\therefore 15x \cdot 12 \cdot 60$ = entire discharge in 12 hours

$\therefore 15x \cdot 720 = 720 \cdot 2(6x + 1) + 720$
$15x = 12x + 3$
$3x = 3$
$x = 1$

$\therefore$ 1st discharged 3 gallons, 2d discharged 2 gallons at one stroke
Water introduced by pipe in one minute = $(720 \cdot (6x + 1))/(6 \cdot 60) = 2 \cdot 7 = 14$ gallons

Redux

50. In a mixture of wine and brandy, half of the whole plus 15 gallons was wine and one-third of the whole minus 3 gallons was brandy; how many gallons were there of each.

Let $6x$ = the whole amount which gives you integer coefficients for the half and third.

Answer: 51 gallons of wine and 21 gallons of brandy.

51. From each of 16 equal coins an artist filed the worth of half-a-crown (30d.), and then offered them in payment for their original value; but being detected, the pieces were found to be really worth no more than 12 guineas in all; what was their original value? (1 guinea = 21s.)

Let x = number of sixpences each was originally worth.

Answer: x = 18s. 3d.

52. A has L.600, and B L.460; if A increases his capital by L.4 per month, and B by L.1 per month, in how many months will A be twice as rich as B?

Let x = the months needed and calculate A's and B's increases from that.

Answer: x = 160 months

53. A besieged garrison had such a quantity of bread as would, if distributed to each man at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces a day for 8 weeks; required the original number of men in the garrison.

Let x = the original number of men
Then 7 days × 6 weeks × 10x ounces is the amount of bread

Answer: 3200 men.

n = x + n - x

Remember when we bought 12 yards of silk, some at 19s. and the rest at 17s., we bought x at 19s. and the rest had to be 12 - x. For every number n, n = x + n - x, which is very useful when we need to know what x is. And it gives us a place to start.

54. Divide the number 197 into two such parts that 4 times the greater may exceed 5 times the lesser by 50.

Let x = the lesser

∴ 197 - x = the greater

∴4(197 - x) = 5x + 50
788 - 4x = 5x + 50
738 = 9x
82 = x = lesser

∴ 115 = the greater

Now try the same problem by making 197 - x the lesser and x the greater. Isn't that weird? Conceptually, it is a bit weird. But when you think about it, what determines things are the relations: the four times, the five times, the excess. The actual values must obey the relations no matter how you set it up -- if you set it up correctly.

55. Divide the number 68 into two such parts, the the difference between the greater and 84 may equal three times the difference between the less and 40.

Let x = the less

∴ 68 - x = the greater

∴ 84 - (68 - x) = 3(40 - x)
16 + x = 120 - 3x
4x = 104
x = 26

∴ the greater = 42

56. A man at a party betted three shillings to two upon every deal. After 20 deals, he won 5s. How many deals did he win?

Let x = number of deals won

∴ 20 - x = number of deals lost

Now he is betting 3s. to 2s. So if he loses, he loses 3s.
∴ he wins 2x and loses 3(20 - x)
∴ 2x - 3(20 - x) = 5 = the 5s. he won
5x = 65
x = 13

∴ He won 13 deals.

57. A Gentleman employed two labourers at different times, one for 3s. and the other for 5s. a day. Now the number of days added together was 40 and they each received the same sum. How many days was each employed?

Let x = number of days 2d employed

∴ 40 - x = number of days 1st employed

∴ 5x = 3(40 - x)
x = 15

∴ 1st employed 25 days and 2d employed 15 days

58. A Field of wheat and oats which contained 20 acres was put out to a labourer to reap for six guineas, the wheat at 7 shillings an acre, and the oats at 5 shillings. Now the labourer falling ill, reaped only the wheat. How much money ought he to receive according to the bargain? A guinea is 21s.

Let x = acres of wheat

∴ 20 - x = acres of oats
∴ 7x = price of reaping wheat in shillings
∴ 5(20 - x) = price of reaping oats

∴ 7x + 100 - 5x = 6·21 = 126
2x = 26
x = 13

∴ 7·13s. = 91s. = **L**.4. 11s. is his pay

59. It is required to divide the number 91 into two such parts that the greater, divided by the difference, the quotient may be 7.

Let x = greater

∴ 91 - x = lesser

∴ x/(2x - 91) = 7
x = 14x - 637
13x = 637
x = 49

∴ The parts are 49 and 42

60. If 10 apples cost a penny, and 25 pears cost two-pence, and I buy 100 apples and pears for nine-pence half-penny, how many of each shall I have?

Let x = number of apples

∴ 100 - x = number of pears

∴ x/10 + (100 - x)/25 = 9.5 What does this line **mean**?
25x + 20(100 - x) = 2375
25x - 20x + 2000 = 2375
5x = 375
x = 75

∴ 75 apples and 25 pears

61. A person has two sorts of wine, one worth 20d. a quart, and the other 12d; from both of which he would mix a quart to be worth 14d. How much of each must he take?

Let x = amount of 20d.

Let the whole quart = 1
∴ 1 - x = amount of 12d.
20x = value of the first
12 - 12x = value of the second

∴ 20x + 12 - 12x = 14
8x = 2
x = 1/4

∴ He must use 1/4qt. of 30d. and 3/4qt. of the 12d.

Redux

62. Bought 24 yards of cloth for **L**.21. 8s. For part of it I paid 19s. a yard and 17s. a yard for the rest; how many yards of each were bought.

Let x = yards at 19s.

Change the money to shillings. That will equal the sum of the yards times their cost.

Answer: 10 yards at 19s. and 14 at 17s.

63. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

Remember that x can be the greater or the lesser.

If A exceeds B by C, how do you set up an equality with A on one side and B on the other? Use tiny numbers to find out.

Answer: 115, 82

64. Divide the number 68 into two parts so that 84 diminished by the greater may be equal to three times 40 diminished by the lesser.

Let x = the less so that you can write 3(40 - x) which is simpler than 3(40 - (68 - x))

Answer: 26, 42

Fraction X

When the values are fractional and they don't lend themselves to using LCM, it is easier to just go with the fractions and put up with the extra arithmetic. This really isn't its own form of solution. Here we are only looking at different ways to handle fractions.

65. Upon measuring the corn produced by a field of 48 quarters; it appeared that it yielded only one-third part more than was sown. How much was that?

Let 4/3 x = what was produced.
This comes from 3/3 being all of what was sown and you got one-third more.

∴ 48 = 4/3·x This line is actually the answer in this case. You could stop here.
36 = x

∴ The quantity sown was 48 quarters (whatever those are …)

You could also do this as 3x = quarters sown then 3x + x = 48 then x = 12 and 3x = 36.

66. A farmer sold 96 loads of hay to two persons. To the first one-half and to the second one-fourth of what his stack contained. How many loads did that stack contain?

Let 4x = the number of loads Why 4x?

∴ 2x = what 1st bought
∴ x = what 2d bought
∴ 2x + x = 3x = 96
x = 32

∴ Stack contained 4·32 = 128 loads

This could also be solved as 3/4·x = 96.

67. A Gentleman bequeathed **L**.210 to two servants; to one he left half as much as to the other. What did he leave to each?

Let 3/2·x = the whole and half shares

∴ 3/2·x = 210
∴ x = 140 and the other servant got 70

Or let 2x = the greater and x = the lesser
∴ 3x = 210 ∴ x = 70 and the greater was 140

68. A Gentleman gave to 3 persons 98 pounds. The second received five-eighths the sum of the first and the third one-fifth of the second. What did each receive?

Let 8x = amount of first person

∴ 5x = amount of second person
∴ x = amount of third person

∴ (8x + 5x + x) = 14x = 98
x = 7

∴ Amounts are 56, 35, and 7 pounds.

You could also set this up as x + 5/8·x + 1/5·5/8·x = 98

69. A and B began to pay their debts. A's money was at first two-thirds of B's; but after A had paid one pound less than two-thirds of his money and B one pound more than 7/8 of his, it was found that B had only half as much as A had left. What sum had each at first?

From A : B : 2/3 : 1 :: 2 : 3 so let 2x and 3x be A's and B's starting sums

After payment:
2x/3 + 1 = A's remaining total
3x/8 - 1 = B's remaining total

∴ 2x/2 + 1 = 3x/4 - 2
8x + 12 = 9x - 24
x = 36

∴ A had 72 pounds and B had 108

70. It is required to divide the number 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other.

Let 2x = the first part

∴ x = ⅓ 2d part ∴ 3x = 2d part
x = ¼ 3d part ∴ 4x = 3d part

∴ 2x + 3x + 4x = 9x = 36
x = 4

∴ Parts are 8, 12, 16

[cont'd next page]

And that method is okay if things stay simple.
Another more general way is this:

Three parts: a,b,c: $a/2 = b/3 = c/4$ and $a + b + c = 36$
Now we set everything in terms of a
$\therefore\ b = 3\cdot(a/2) = 3/2\cdot a$ and $c = 4\cdot(a/2) = 2a$
Now let x = a

$$x + 3/2x + 2x = 36$$
$$2x + 3x + 4x = 72$$
$$9x = 72$$

$\therefore x = a = 8$ $b = 3/2\cdot 8 = 12$ $c = 2\cdot 8 = 16$
Next time, save the x=a and just solve for a.

71. Divide the number 116 into four such parts, that if the first be increased by 5, the second diminished by four, the third multiplied by three, and the fourth divided by 4 the result in each case shall be the same. We'll do this the same two ways as the last.

Let x = the third
$\therefore$ 3x = half the fourth
6x = the fourth
3x + 4 = the second
3x - 5 = the first

$$\therefore\ 3x - 5 + 3x + 4 + x + 6x = 116$$
$$13x = 117$$
$$x = 9$$

$\therefore$ parts are 22, 31, 9, 54

OR

$a + b + c + d = 116$
$a+5 = b-4 = 3c = d/2$
Put all in terms of a:
$b = a + 9$ $c = (a+5)/3$ $d = 2a + 10$
$\therefore a + a + 9 + (a+5)/3 + 2a + 10 = 116$

Solve for a and you get 22 and the rest follows.
For me this second way is more straightforward and the first takes too much thought. But your mileage may vary.

72. A poor man with a wife and seven children, found during a scarcity that he could only earn sufficient to procure ¼ of a white loaf of bread per day for each of his family, himself included. He therefore applied to the parish officers for assistance, by whom being allowed a daily sum = ½ his earnings, and mixed bread being made by order of Parliament, which was cheaper than white in proportion of 4 to 5, he was now enabled to procure ⅓ of a mixed loaf per day for each of the family (himself still included) and had 1s. 7½d. over. Required the sum allowed him by the parish.

Let x = price of white loaf in pence

∴ $9x/4$ = what he earned
$9x/8$ = what parish allowed him
$4x/5$ = price of mixed loaf

$$\therefore \quad 12x/5 + 39/2 = 9x/4 + 9x/8 = 27x/8$$
$$96x + 780 = 135x$$
$$39x = 780$$
$$x = 20$$

∴ He earned 45d. and had 22½d. from the parish.

Redux

73. What number is that whose third part exceeds its fourth part by 16?

Let 12x = the number so it has integer coefficients for its third and fourth parts.

Answer: 192

74. A sum of money was divided between two persons, A and B: A's share exceeded 5/9 of the whole by **L**.50 and A's share was to B's as 5 to 3; what was the share of each?

Let 5x = A's share. You know what B's is. Use 5/9 of their total to get x.

Answer: A's and B's shares were 450 and 270 pounds.

Ratios

When we represent two numbers, a,b, as a:b we are using Euclid's ratios. And when we have a:b::c:d, Euclid is saying "a is to b as c is to d". Now his ratios and proportions (that last thing) are not strictly fractions, as you will see. But fractions using numbers fall under all of the laws of ratio and proportion. So we can treat a:b::c:d as if it meant a/b = c/d and we won't go wrong.

Modern education tends to ignore ratio and proportion. It completely ignores Euclid. All this ignorance is stupid. Ratio and proportion are powerful tools. Set them up as fractions as you work through the problems so you can see what is going on from step to step. More about this as we go along.

75. A prize of **L**.2329 was divided between two persons A and B, whose shares therein were in proportion of 5 to 12. What was the share of each?

Here Dr. Bland means "in **ratio** of 5 to 12".

Let 17x = all the parts (5 + 12)

∴ 17x = 2329
x = 137

∴ A got 5·137 = 685 and B got 12·137 = 1644

So where is the ratio or proportion in all this?
You can work out for yourself that if a:b::c:d then bc = ad.
So in this problem we have A:B::5:12 and a total of 2329.

∴ 12A = 5B ∴ B = 12/5·A
∴ A + B = 5/5 · A+ 12/5 · A = 17/5·A = 2329
∴ So A got 685 and you can figure out how to calculate for B.

On the top, we just calculated things by the ratio. Below, we treated it as a proportion. Some problems come in the form of a proportion and the simpler way isn't available.

76. A sum of money is to be shared between two persons A and B, so that as often as A receives 9 pounds, B takes 4. Now it happens that A receives 15 pounds more than B. What are their respective shares?

Let x = B's share, the simplest data here.

You can solve this by thinking in terms of a proportion:

$9 : 4 :: x + 15 : x$
$\therefore 9x = 4x + 60$
$5x = 60$
$x = 12$ is B's share

∴ 27 is A's share.

Or you could let 9x = A's share and 4x = B's share

$\therefore 9x = 4x + 15$ or $5x = 15$ or $x = 3$

A gets 9x = 27 and B gets 4x = 12

77. What two numbers are as 2 to 3; to each of which if 4 be added, the sums will be as 5 to 7?

Let the numbers be 2x and 3x

$\therefore 2x + 4 : 3x + 4 :: 5 : 7$

$a:b::c:d \Rightarrow ad = bc$

$\therefore 14x + 28 = 15x + 20$
$x = 8$

∴ Numbers are 16 and 24.

78. A joint stock of two partners whose particular shares differed by **L**.40 was to the share of the lesser as 14 to 5. Required the shares.

Let 14x = joint stock

∴ 5x = lesser
9x = greater

∴ 9x = 5x + 40
4x = 40
x = 10

∴ shares are 90 and 50 pounds

Or 2x - 40 : x - 40 :: 14 : 5

Here total = x + x - 40 = 2x - 40
So 10x - 200 = 14x - 560

∴ x = 90 = A's share and 90 - 40 = 50 = B's share

The first method uses the ratio directly in the variable coefficients.
The second uses it in a proportion with the total and B's share.

79. A Bankrupt owed to two Creditors 140 pounds; the difference of the debts was to the greater as 4 to 9. What were the debts?

Let 4x = the difference

∴ 9x = greater
∴ 5x = the lesser

∴ (9x + 5x) = 14x = 140
x = 10

∴ debts were 90 and 50 pounds

Or we can do A + B = 140 and A : B :: x : 140 - x

∴ A - B = 2x - 140
∴ A - B : A :: 4 : 9
∴ 2x - 40 : x :: 4 : 9

Use a:b::c:d ⇒ ad = bc and algebrate.

80. A and B made a joint stock of **L**.833, which, after a successful speculation, produced a clear gain of **L**.153. Of this B had **L**.45 more than A. What did each person contribute to the stock?

In joint stock, each receives a proportion of the gain equal to the proportion he or she puts in.

Let x = B's investment

∴ 833 : x :: 153 : 9/49·x where 153/833 = 9/49
∴ 9/49·x - 45 = A's gain

∴ 18x/49 - 45 = 153
18x/49 = 198
x = (49·198)/18 = 49·11 = 539

∴ B invested **L**.539 and A **L**.294

If you try ad = bc here you get 9/49·833x = 153x or 153x = 153x which is no help.

You need, geometrically speaking, the - 45 to give you a line with A that sums with B's line through the origin to equal the given gain. Then you have something to solve.

81. Sold a pound of tobacco for 19s., part at one shilling a pound and the rest at 15d. Now the first part to the latter :: 3/4 : 2/3. How much was sold of each?

We can first work with the ratio 3/4 : 2/3 :: 9 : 8 from 3/4 = 2/3.

∴ 9x = lbs. of the former

∴ 8x = lbs. of the latter
9x = number shillings the first sold for

∴ 8x · 5/4 = number shillings the latter sold for (from 15d. = 12d + 3d = 5/4s.)

∴ 10x + 9x + 19x = 19
x = 1

∴ 9lbs at 1s. and 8lbs. at 15d.

82. Suppose that for every 10 sheep a farmer kept, he should plough an acre of land, and be allowed one acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres?

Let x = number of sheep

Now the number of acres ploughed per sheep is x/10 and the pasture x/4
(from 10 : x :: 1 : ploughed and 4 : x :: 1 : pasture)

$\therefore x/10 + x/4 = 700$
$2x + 5x = 700 \cdot 20$
$x = 20 \cdot 100 = 2000$ sheep

This is an example of how ratio can be used to establish values in your model.

83. A person was asked the hour and answered that it was between five and six; and the hour- and minute-hands were together. What was the time?

Let x = time past 5

Minute hand goes 12 times for every hours 1 time
$\therefore 12 : 1 :: 5+x : x$ where 5+x is five o'clock plus x minutes

Now if you really knew proportions, you would know that this proportion equals
11 : 1 :: 5 : x. But you will have to take the long way around with ad = bc.

$\therefore 12x = 5 + x$
$11x = 5$
$x = 5/11$ of an hour

∴ Time is 5 o'clock and 27.27 minutes.

For the stalwart-hearted, you could also use

$$12 : 1 :: x : x - 5/12$$

and all of these are proportions are minutes : hours :: minutes : hours

84. Divide the number 49 into two parts such that the greater increased by 6 may be to the lesser diminished by 11 as 9 to 2.

Let x = greater part

∴ 49 - x = lesser part

∴ x+6 : 38-x :: 9 : 2 What is the 38 - x?

Use the numbers from 9:2 to figure out what is going on here.
This is the power of proportions.

∴ x+6 : 44 :: 9 : 11
x+6 : 4 :: 9 : 1

∴ x + 6 = 36
x = 30

∴ Parts are 30 and 19.

Again, in order to make sense of proportions, lay them out as pairs of equal fractions. Then figure out what was done to each side. Proportions as a/b = c/d are homogeneous expressions. So whatever you do with one side, you can do with the other. For instance if a : b :: c : d then a+b : b :: c + d : d or (a+b)/b = (c+d)/d. Likewise you can multiply or divide numerator or denominator by constants. Just look at them and learn.

85. A, B, and C make a joint stock; A puts in **L**.60 less than B and **L**.68 more than C and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in?

Let x = A's share

∴ x + 60 = B's share
x - 68 = C's share

∴ 2x + 60 : 2x - 8 :: 5 : 4 What does this line **mean**?
x+30 : x-4 :: 5 : 4
34 : x-4 :: 1 : 4

∴ 136 = x - 4
x = 140

∴ A, B, C put in 140, 200 and 72 pounds respectively.

86. It is required to divide the number 34 into two such parts, that the difference between the greater and 18 shall be to the difference between 18 and the lesser as 2:3.

Let x = greater

∴ 34 - x = lesser

x - 18 : x - 16 :: 2 : 3 (figure out the x - 16)
x - 18 : 2 :: 2 : 1 (3 - 2 = 1 is symmetrically where the 1st two comes from)

∴ x - 18 = 4
x = 22

∴ Parts are 22 and 12.

87. A Bookseller sells two books, one containing 100 sheets for 10s., the other containing 50 sheets for 6s., each being bound at the same price. What was that price?

Let x = price

∴ 10 - x : 6 - x :: 100 : 50 (by extension) :: 2 : 1
∴ 4 : 6 - x :: 1 : 1
∴ 4 = 6 - x

x = 2 = price of binding in shillings

We can always extend a proportion. Seeing the 100 : 50, we reduce it to 2 : 1, and use the 2 :1 from there.

88. A man wished to enclose a piece of ground with palisadoes, and found that if he set them a foot asunder, he should have too few by 150; but if he set them a yard asunder, he should have too many by 70. How many palisadoes had he?

Let x = number of palisadoes

∴ x - 70 :: x + 150 :: 1 : 3 where the last two are per yard
x - 70 : 220 :: 1 : 2
x - 70 : 110 :: 1 : 1
∴x - 70 = 110

∴ x = 180 = number of palisadoes.

Can you tell I like the word "palisadoes"?

89. A Footman, who contracted for **L**.8 a year, and a livery suit, was turned away at the end of 7 months, and received only **L**.2. 3s. 4d. and his livery. What was its value in pounds?

Let x = livery value in pounds

$\therefore$ 12 : 7 :: 8 + x : 13/6 + x :: 48 + 6x : 13 + 6x
Can you justify the 13/6? And what about the extension: 48 + 6x : 13 + 6x?
$\therefore$ 5 : 7 :: 35 : 13 + 6x
1 : 7 :: 7 : 13 + 6x
$\therefore$ 13 + 6x = 49
6x = 36

$\therefore$ x = 6 = livery's worth in pounds

90. What number is that to which if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third?

Let x = number required

$\therefore$ x + 1 : x + 5 :: x + 5 : x + 15
x + 1 : 4 :: x + 5 : 8
x + 1 : x + 5 :: 4 : 8 :: 1 : 2
x + 1 : 4 :: 1 : 1
x + 1 = 4

$\therefore$ x = 3

91. A Market-woman bought a certain number of eggs at two a penny, and as many at three a penny, and sold them at a rate of five for two-pence; after which she found that instead of making her money again as she expected, she lost four pence by them. How many eggs of each sort had she?

Let x = number of eggs of each sort (both sorts equal)

$\therefore$ 2 : x :: 1 : x/2 (or 2d. : x :: 1d. : x/2) = price of eggs at two a penny
Sym. x/3 = price of eggs at three a penny (3 : x :: 1 : x/3)
Adding the first three terms and using the 5 for 2d in the fourth.
5 : 2x :: 2 : 4x/5

$\therefore$ 4x/5 + 4 = x/2 + x/3 So what does this line **mean**?
$\therefore$ 24x + 120 = 15x + 25x

$\therefore$ x = 120 eggs of each kind

You will often have occasion to use the above idea of n : x :: m : n/x. Be sure you understand what this is saying in the problem above.

92. A man and his wife did usually drink out a vessel of beer in 12 days: but when the man was out, the vessel lasted the woman 30 days. In how many days would the man alone empty the vessel?

Let x = number of days required

∴ x : 12 :: 1 : 12/x = part drunk by man in 12 days
30 : 12 :: 1 : 12/30 = 2/5 = part drunk by woman in 12 days

∴ 12/x + 2/5 = 1
60 + 2x = 5x
60 = 3x
20 = x

This idea of parts adding up to unity is important.

93. A cistern into which water was let by 2 cocks A and B, will be filled by them both running together in 12 hours, and by cock A alone in 20 hours. In what time will it be filled by cock B alone?

Let x = number of B's hours

x : 12 :: 1 : 12/x = quantity from B in 12 hours
Sym. A supplies 3/5 in 12 hours

∴ 12/x + 3/5 = 1
60 + 3x = 5x
60 = 2x
x = 30

Don't let me confuse you here. But there is another method. A does 12 together and 20 alone. So we have 12/20 and therefore 12/20 + 8/20 = 1. So ratio of B : A is 8 : 12. Then 8/12 = 20/x. This last actually says something and works out to x = 30. If you understand its meaning, then use it. If not, leave it alone until you do. It is simply a quicker method.

94. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, but the less three times in two minutes; and the whole time of emptying was 12 minutes. Required the size of each.

Let x = number of gallons in lesser bucket

∴ 2x = gallons in greater
∴ 4x = thrown out by greater in 3 minutes
∴ 3 : 12 :: 1 : 4 :: 4x : 16x = quantity thrown by greater out in 12 minutes
Sym. in 12 minutes, lesser throws out 18x

∴ 18x + 16x = 34x = 442
x = 13

∴ gallons in buckets are 13 and 26

You could also set this up as

$$2x \cdot 2/3 \cdot 12 + x \cdot 3/2 \cdot 12 = 442$$

which again is faster **if you know what you are meaning by it**.

95. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound make if the greyhound would take hare?

I hate these leapy problems. Here's the most painless approach:

Let 3x = number of leaps greyhound must take

∴ 4x = leaps made by hare in that time
∴ 4x + 50 = total hare leaps (50 being its lead on the dog)
∴ 2 : 3 :: 3x : 4x + 50

∴ 9x = 8x + 100
x = 100

∴ 3x = 300 leaps for greyhound to take hare

If you left out the 50, you solve for when the dog reaches where the hare is at the beginning of the race. But you want the end of the race. Also, think about what x actually means here.

96. A person engaged to reap a field of 35 acres, consisting partly of wheat and partly of rye. For every acre of rye he received 5s.; and what he received for an acre of wheat augmented by 1s. is to what he received for the rye as 7 to 3. For his whole labour he received **L**.13. Required the number of acres of each sort.

Let x = acres of wheat

∴ 35 - x = acres of rye
∴ 175 - 5x = price of his reaping the rye

3 : 7 :: 5 : 1 + price of reaping acre of wheat = 35/3
∴ price of reaping acre of wheat = 35/3 - 1 = 32/3
∴ 32x/3 = price of reaping all the wheat

∴ 32x/3 + 175 - 5x = 260 (= 13 pounds in pence)
17x = 255
x = 15

∴ 15 acres of wheat and 20 of rye

97. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one of **L**.5, the other for **L**.6. 10s. Now if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece and how much did they cost a yard?

Let x = number of 10 shillings each yard cost

∴ 10/x = length of the least
13/x = length of the greatest

Here, a pound is 20s. or 2×10s.
So the short comes to 10×10s. and the long comes to 13×10s.

∴ 10/x + 10 : 13/x + 10 :: 5 : 6
10(1/x + 1) : 3/x :: 5 : 1 and this is a : b-a :: c : d-c from a : b :: c : d
2(1/x + 1) : 3/x :: 1 : 1

∴ 2((x+1)/x = 3/x
2x + 2 = 3
2x = 1
x = ½

∴ Price per yard is 5s. and the lengths are 20 and 26 yards

98. Before noon, a clock which is too fast, and points to afternoon time is put back five hours and forty minutes; and it is observed that the time before shown is to the true time as 29:105. Required the true time.

Let x = time first pointed to

∴ x : x + 6⅓ :: 29 : 105 Why the +6⅓?
x : 19/3 :: 29 : 76
x : ⅓ : 29 : 4

∴ x = 29/12 = 2 hrs 25 min

∴ add this to the 6 hr 20 min (6⅓) and the true time is 8:45

Again, if you can understand every algebraic statement and every value in the solution, you will have learned a great deal. Studying solutions is a big part of our experience in mathematics.

99. A Merchant buys a cask of brandy for **L**.48, and sells a quantity exceeding three-fourths of the whole by 2 gallons at a profit of 25%. He afterwards sells the remainder at such a price as to clear 60% by the whole transaction; and, had he sold the whole quantity at the latter price, he would have gained 175%. Required the number of gallons contained by the cask.

Let 4x = number of gallons Why 4x? Two possible answers.

∴ 12/x = original price per gallon in pounds

∴ 100 : 125 :: 12/x : 15/x = first sale because 125% of 12 = 15
100 : 275 :: 12/x : 33/x = second sale for same reason

∴ (3x + 2)·15/x + (x-2)·33/x = whole gain = 30 - 36/x
∴ 100 : 60 :: 48 : 30 - 36/x
5 : 3 :: 8 : 5 - 6/x

24 = 25 - 30/x
30/x = 1
x = 30

∴ 4x = 120 gallons

Redux

100. A and B make a joint-stock of **L**.870, which , after a successful speculation, produced a clear gain of **L**.174. Of this A received **L**.36 more than B; what did each contribute to the stock?

Let x = sum A contributed to the stock

You need a proportion of:

total invested : A's investment :: total gain : A's gain

to get a term for A's gain which is used to compute B's gain as a term

Answer: A and B contributed 525 and 345 respectively.

101. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 5 as 7 to 8.

Let x = the less

Answer: The parts are 20 and 29.

Diagrams

If you can draw a diagram for anything you are modelling, the diagram will help. But sometimes a diagram is essential. By drawing the diagram and filling it in with the data, you show yourself the relations that must exist in the model. Sometimes diagrams are pictures like in this first example. Sometimes they are lists of algebraic data. In both cases you can discern the relations by laying out the diagram.

102. Four places are situated in the order of the four letters A, B, C, D. The distance from A to D is 34 miles, AB : CD :: 2 : 3, and ¼AB + ½CD = 3BC. What are the respective distances?

Let 2x = AB

∴ 3x = CD
∴ x/2 + 3x/2 = ¼AB + ½CD = 2x = 3BC
∴ BC = 2/3·x
∴ 2x + 3x + 2/3·x = 17/3·x = 34
∴ x/3 = 2
∴ x = 6

∴ AB = 12, BC = 4, and CD = 18

You could also set this up as
2x = AB and 3x = CD
∴ BC = 34 - 5x
∴ ¼·2x + ½·3x = 3(34 - 5x) and algebrate

In the first version, the model is AB+BC+CD=34 and the second is ¼AB+½CD = 3BC.
But all you need is anything **true** that will give you x.

103. A person at play won twice as much as he began with, and then lost 16s. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money found he had 80s. What sum did he begin with?

I call this kind of algebraic diagram a **trace**. You list what goes on in an algebraic way and then use the list to build your equation.

x	began with
+2x	won
-16s	loss
-4/5(total)	lost 4/5 of total
+ x	won beginning amount
80s.	total at end

∴ x + 2x - 16 - 4/5(x+2x-16) + x = 80
And this resolves down to 8x = 416
∴ x = 52

You can also look at it this way
Take it to 3x - 16 above. If he loses 4/5, he has 1/5 or (3x-16)/5
And (3x-16)/5 + x = 80 and again x = 52.

If your trace is correct, anything you correctly build with it will work. You will make fewer mistakes if you begin by building these the long way and gradually develop your own methods of shortening them.

104. A Shepherd, in time of war, was plundered by a party of soldiers, who took ¼ of his flock, and ¼ of a sheep, another party took from him ⅓ of what he had left and ⅓ of a sheep; then a third party to ½ of what now remained and ½ of a sheep. After which he had but 25 sheep left. How many had he at first?

Let x = number of sheep at first

Here we build the trace:
∴ (x + 1)/4 = number first taken
∴ (3x - 1)/4 = number remaining
After second party takes their thirds
2/3((3x - 1)/4) - ⅓ = (x-1)/2 = remaining sheep
After third party take their halves
∴ (x - 1)/4 - ½ = (x - 3)/4 remain

∴ (x - 3)/4 = 25
x - 3 = 100
x = 103

105. A man being at play lost ¼ of his money, and then he won 3s.; after which he lost ⅓ of what he had, and won 2s.; lastly he lost 1/7 of what he then had; this done he had but 12s. left. What had he at first?

Let 4x = total at first in shillings Why 4x?

Build a trace:
3x = after 1st loss
3x + 3 = after he won 3s.
2/3(3x + 3) = 2x + 2 = after 2d loss
2x + 4 = after won 2s.
6/7(2x + 4) = (12x + 24)/7 = after final loss

∴ (12x + 24)/7 = 12
x = 5

∴ He began with 20s.

106. A Trader maintained himself for 3 years at the expense of **L**.50 a year; and in each of those years, augmented by that part of the stock which was not so expended by one-third thereof. At the end of the third year his original stock was doubled. What was that stock in pounds?

Let x = number of pounds required

Build a trace
∴ x - 50 = sum not expended, with which he traded
∴ (x - 50)/3 = 1st year gain
∴ 4/3·(x - 50) = sum he had at end of 1st year
∴ (4x - 200)/3 - 50 = (4x - 350)/3 = sum he traded with in 2d year
∴ 4/3·(4x - 350)/3 = (16x - 1400)/9 = sum at end 2d year
∴ (16x - 1400)/9 - 50 = (16x - 1850)/9 = sum traded with in 3d year
∴4/3·(16x - 1850)/9 = sum at end of 3d year

∴ 4/3·(16x - 1850)/9 = 2x
32 - 3700 = 27x
5x = 3700

∴ x = 740

The Bendy Line

A quadratic equation is simply a uniformly bent line. It takes the form $ax^2 + bx + c$. And it has only two factors. We are going to look at simple states of affairs which can be modelled by a simple quadratic equation. To solve these, you will need to know the "quadratic equation" which is a stupid name for the solution to all quadratics:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Once you have the quadratic, you simply plug a, b, and c into that formula and you have the two values of x which make $f(x) = ax^2 + bx + c = 0$. These are the roots. In some cases, only one root can be the solution. If you want to know how much wheat there is, the wheat can neither be negative wheat nor imaginary wheat. But the roots can sometimes be either or both of these.

Simple Quadratics

107. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

Let $10x$ = their sum

$\therefore$ $7x$ and $3x$ = the greater and lesser numbers (Both taken from 10:7)

$\therefore 30x^2 = 270$
$x^2 = 9$
$x = \pm 3$

The numbers are 21, 9 and -21,-9.

Here 30 : 21 :: -30 : -21 :: 10 : 7. So the negative values work.

108. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are those numbers?

Let $4x$ and $5x$ = the numbers

$\therefore 25x^2 - 16x^2 = 9x^2 = 81$
$x^2 = 9$
$x = \pm 3$

The numbers are ±12 and ±15. When we have numbers like this with the ± sign, it indicates that we can take ++ or --. If we want to take them with opposite signs, we use: ±12 and ∓15.

These simple quadratics where you have $ax^2 = c$ are called pure quadratics. And you don't need to use the "quadratic equation" to solve them. You just need the positive and maybe the negative square root of c/a.

109. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Let 2x = their difference

∴ 9x = the greater
∴ 7x = the lesser

∴ $81x^2 - 49x^2 = 32x^2 = 128$
$x^2 = 4$
$x = \pm 2$

The numbers are ±18 and ±14.

110. A Mercer bought a piece of silk for **L**.16. 4s.; and the number of shillings which he paid for a yard was to the number of yards as 4:9. How many yards did he buy and what was the price of a yard?

Let 4x = number of shillings paid per yard

∴ 9x = number of yards
∴ $36x^2 = 324$ (324 = what he paid in shillings)
$x^2 = 9$
$x = \pm 3$ (Can't use the -3.)

Consequently, there were 27 yards, at 12s. per yard.

111. Find three numbers in the proportion of 1/2, 2/3, and 3/4; the sum of whose squares is 724. (What do we even do with the proportion thing? Well, if we put them over a common denominator, the numerators are in the same proportion and are 6, 8, and 9. Try it and see.)

∴ 6x, 8x, 9x = the numbers

∴ $36x^2 + 64x^2 + 81x^2 = 181x^2 = 724$
∴ $x^2 = 4$
$x = \pm 2$

The numbers are ±12, ±16, and ±18.

112. What two numbers are those whose difference is to the less as 4 to 3; and their product multiplied by the less is equal to 504?

Let 4x = the difference
and 3x = the lesser number

∴ 7x = the greater number

∴ $63x^3 = 504$
$x^3 = 8$
$x = 2$

And the numbers are 6 and 14.

113. What two numbers are as 5 to 4, the sum of whose cubes is 5103?

Let 5x, 4x = the numbers

∴ $125x^3 + 64^3 = 189x^3 = 5103$
$x^3 = 27$
$x = 3$

And the numbers are 15 and 12.

114. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as there were boys. They filled their bags, and found the number of apples was 2916. (Good job, orchard thieves!) How many boys were there?

Let x = number of boys

∴ x^2 = number of bags
∴ $4x^3$ = number of apples

∴ $4x^3 = 2916$
$x^3 = 729$
$x = 9$

There were 9 boys. So what do 9 boys do with 2916 apples?

115. A person bought for one crown (= 5s.) as many pounds of sugar, as were equal to half the number of crowns he laid out. In selling the sugar he received for every 25lbs. of it as many crowns as the whole had cost him; and he received on the whole 20 crowns. How many crowns did he lay out, and what did he give for a pound?

Let x = number of crowns laid out

∴ $x/2$ = number of pounds for one crown
∴ $x^2/2$ = number of pounds in all
∴ $x/25$ = selling price of one pound

∴ $x^2/2 \cdot x/25 = 20$
$x^3 = 1000$
$x = 10$

He laid out 10 crowns and gave one shilling for a pound.

116. A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Required the number of men.

Let x = number of men in front

∴ x + 5 = number in depth

∴ $x^2 + 5x = 5x + 4225$ What does this line **mean**?
$x^2 = 4225$
$x = 65$

Then number of men = 5x + 4225 = 4550

117. A certain sum of money is divided every week among the members of a corporation. It happened one week that the number of members was the square root of the pounds to be divided. Two men, becoming members the next week, diminished the dividend of each of the former members by **L**.1 6s. 8d. What was the sum?

Let x^2 = sum in pounds

∴ x = number of members and also the sum each received

∴ $x - 4/3 = x^2/(x+2)$ What does this line **mean**?
∴ $x^2 + (2/3)x - 8/3 = x^2$
$2/3 \cdot x = 8/3$
$x = 4$

∴ x^2 = 16 pounds divided among members

118. Two workmen A and B were engaged to work for a certain number of days at different rates. And the end of the time, A who had played 4 of those days, had 75s. to receive; but B who had played 7 of those days, received only 48s. Now if B had only played 4 days, and A played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each per day?

Let x = number of days worked

∴ x - 4 = days A worked
x - 7 = days B worked

daily pay:
75/x-4 = A's shillings per day
48/x-7 = B's shillings per day

$$\therefore \frac{75(x-7)}{x-4} = \frac{48(x-4)}{x-7}$$ What does this equality **mean**?

$$25(x-7)^2 = 16(x-4)^2$$
$$5(x-7) = 4(x-4)$$
$$x = 19 \text{ or } 17/3$$

They were engaged to work for 19 days
A worked 15 and B worked 12.
A received 5s. per day and B 4s.

And what happens if you use 17/3 days?

Note

We'll put all the pure quadratic reduxes, plain or with ratios, with the impure.

Simple Ratio and Proportion

In older books, if a:b::c:d, we know that ad = bc. And in newer books, if a/b = c/d, we still have ad = bc. And under either of these, lie Euclid's ratios and proportions. So if there is an x in both a and d or both b and c, we end up with an x^2 and our representation of the data becomes a quadratic.

119. It is required to divide the number 18 into two such parts, that the squares of those parts may be in the proportion of 25 to 16.

Let x = greater part

∴ 18 -x = lesser part

∴ $x^2 : (18 - x)^2 :: 25 : 16$ (Which could be written $x^2/(18-x)^2 = 25/16$)
x : 18 - x :: 5 : 4
x : 18 :: 5 : 9 (How did we get this?)
x : 2 :: 5 : 1 (And this?)

∴ x = 10

The parts are 10 and 8.

Doing it this way: $x^2/(18-x)^2 = 25/16$
You can still get: x/18-x = 5/4 because we can't use the negative square roots anyway
Then you have:

4x = 5(18-x)
4x = 90 - 5x
9x = 90
x = 10

But if you will take the time to learn the older way, it is much more powerful for handling more complex relations. But I'm not your mom. Do as you like.

120. It is required to divide the number 14 into two such parts that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater as 16:9.

Let x = the greater

∴ 14 - x = lesser
∴ x/(14-x) : (14-x)/x :: 16 : 9
$x^2 : (14-x)^2 :: 16 : 9$
x : 14-x :: 4 : 3 ∴ x : 14 :: 4 : 7 ∴ x : 2 :: 4 : 1 ∴ x = 8

And the parts are 8 and 6.

121. A number of shillings were placed at equal distances on a table so as to form the sides of an equilateral triangle; then from the middle of each side a number of shillings, equal to the square root of the number in the side were taken, and placed upon the corner shilling opposite to that side; it then appeared that the number on each side was to the number previously upon it, as 5 to 4. Required the number of shillings on one side at first.

This problem begs the question: "What is this person doing?"
But we can't answer that.

Let x = number of shillings on a side

$\therefore x : x - \sqrt{x} :: 5 : 4$
$4x = 5x - 5\sqrt{x}$
$x = 5\sqrt{x}$
$\sqrt{x} = 5$

$\therefore x = 25$ = coins on original side

122. Two partners A and B divided their gain of **L**.60. and B took **L**.20 of this; A's money continued in trade 4 months, and if the number 50 be divided by A's money, the quotient will give the number of months that B's money, which was **L**.100, continued in trade. What was A's money, and how long did B's money continue in trade?

Let x = A's money in pounds

$\therefore 50/x$ = months B's money in trade

Since B gained 20 pounds, A gained 60 - 20 = 40

$\therefore 4x : 5000/x :: 2 : 1$ Where do the factors of 4 and 100 come from?
$x : 2500/x :: 1 : 1$ Use fractions to derive this line from the last one.
$x^2 = 2500$
$x = 50$

A's money was **L**.50, and B's money was one month in trade

In the line $4x : 5000/x :: 2 : 1$ the elements a : b :: c : d
pounds : pounds :: share A : share B
a: A's money x continues to trade for 4 months $\therefore 4x$
b: B's money 100 continues to trade for 50/x months $\therefore 5000/x$
If of 60 pounds, A got 40 and B got 20, the ratio of shares is 2 : 1 which is c : d

Sometimes you try to figure out something like this and you just can't. So let it go for now and move on. You learn something from every problem so long as you actually try to solve it and then really study the solution. Eventually, you will learn enough that you can come back to things you didn't understand and you will easily understand them. You have to give yourself time.

123. Two Travellers A and B set out to meet each other, A leaving town C at the same time B left town D. They travelled the direct road CD, and on meeting it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in 15 and 3/4 days, but B would have been 28 days in performing A's journey. What was the distance between C and D?

Let x = number of miles A travelled

∴ x - 18 = miles B travelled

x - 18 : x :: 15 3/4 : 63x/(4(x - 18)) What does that last term **mean**?

Here we have miles B : miles A :: days B : days A.
And the same below but with A and B swapped:

x - 18 : x :: 28 : (28(x - 18))/x

Take the time to understand the fourth terms in both of these above proportions.

∴ (28(x - 18))/x = 63x/(4(x - 18))
$16(x - 18)^2 = 9x^2$
4(x - 18) = ±3x

∴ x = 72 or 10 2/7

A travelled 72 miles and B travelled 54 ∴ CD = 126

124. A and B lay out some money on speculation. A disposes of his bargain for **L**.11, and gains as much per cent. as B lays out; B's gain is **L**.36, and it appears that A gains four times as much per cent. as B. Required the capital of each.

Let 4x = B's capital and A's gain per cent.

∴ x = B's gain per cent.

100 : 4x :: x : 36
$4x^2 = 3600$
$x^2 = 900$
x = 30

B's capital is 120

∴ 220 : 100 :: 11 : A's capital = (11·10)/ 22 = 5 pounds

125. The Captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings it was found, that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.

You will want to draw a diagram for this one: Let A,B be the original places of the privateer and trader, D the place where the captain changed his course, CE perpendicular to AC. AB = 7, AD = 20.

10 : 8 :: 5 : 4 :: velocity privateer : velocity trader
:: AD : BC :: 20 : BC

$\therefore$ BC = $(20\cdot4)/5 = 16$
$\therefore$ DC = 16 - BD = 16 - 13 = 3
$\therefore$ DE : CE :: 5 : 4 and let CE = x

$\therefore \sqrt{(9+x^2)} : x :: 5 : 4$
$9 + x^2 : x^2 :: 25 : 16$
$9 : x^2 :: 9 : 16$
$x^2 = 16$
$\therefore x = 4$

$\therefore$ DE = 5 $\therefore$ AD+DE = 25 miles travelled by the privateer.

Think of that as a serious lesson in the power of proportions.

126. A Vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

This is not easy. But the model is an important one that you will see again if you go into the sciences.

Let x = gallons drawn 1st time

$\therefore$ 256 - x = wine left
$256 : 256 - x :: x : (x(256 - x))/256$ = quantity drawn 2d time
$\therefore 256 - x - (x(256\text{-}x))/256 = (256\text{-}x)^2/256$ = quantity left after 2d draught
Sym. $(256\text{-}x)^3/256^2$ left after 3d draught
Sym. $(256\text{-}x)^4/256^3$ left after 4th draught
$\therefore (256\text{-}x)^4/256^3 = 81$
And after much strenuous algebration: x = 64

The quantities of wine drawn off each time were 64, 48, 36, and 27 gallons.

Adfected Quadratics

Where the form $ax^2 + bx$ has been called a "pure quadratic", the form $ax^2 + bx + c$ is an "adfected quadratic" in older texts. What this means for you is that the solutions will often require the quadratic equation. I'm going to assume that you are using a calculator to compute the quadratic equation.

127. There are two numbers whose difference is 9, and their sum multiplied by the greater is 266. What are those numbers?

Let x = the greater

$\therefore x - 9$ = the lesser
$\therefore x(2x - 9) = 266$
$2x^2 - 9x = 266$
$2x^2 - 9x - 266 = 0$

$\therefore x = 14, -19/2$
$x - 9 = 5, -37/2$

Since nothing was said about the numbers being integers, you should determine whether the fractions are also a solution.

128. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225.

Let x = the first number

Now you know from the first degree equations with ratios that if

first : second :: second : 16 ⇒ 16 ·first = second^2

$\therefore \sqrt{(16x)}$ = second number
$\therefore x^2 + 16x = 225$
$x^2 + 16x - 225 = 0$
$x = 9, -25$

One solution is clearly 9, 12. What does the -25 give us and is it a solution?

129. A person bought some sheep for **L**.72; and found that if he had bought 6 more for the same money, he would have paid **L**.1 less for each. How many did he buy, and what was the price of each?

Let x = number of sheep bought

∴ 72/x = price of each in pounds
∴ 72/(x+6) = price of each if six more

∴ 72/(x+6) + 1 = 72/x
$72x + x^2 + 6x = 72x + 432$
$x^2 + 6x - 432 = 0$
x = 18, -24 He was not given 24 pounds each to take the sheep, so:

∴ He bought 18 at the price of 72/18 = 4 pounds each.

130. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is equal to that of the glass. Required the width of the frame.

The area of the glass is 12·18 = 216

Let x = width of frame in inches (equal width = same width all around)

∴ area of frame = (18+2x)(12+2x) - 216
∴ (18+2x)(12+2x) -216 = 216
$4x^2 + 60x = 216$
$x^2 + 15x = 54$
$x^2 + 15x - 54 = 0$
x = 3, -18

The width is 3 inches.

131. There are two square buildings, that are paved with stones, a foot square each. The side of one building exceeds the other by 12 feet, and both their pavements contain 2120 stones. What are the lengths of the buildings?

Let x and x+12 = lengths

∴ x^2 and $(x+12)^2$ = number of stones in the squares
∴ $x^2 + x^2 + 24x + 144 = 2120$
$2x^2 + 24x = 1976$
$x^2 + 12x - 988 = 0$
x = 26, -38

The buildings are 26 and 26+12 = 38 feet long.

132. A labourer dug two trenches, one of which was 6 yards longer than the other, for **L**.17. 16s. and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

Let x and x+6 equal number of yards in each

$\therefore x^2 + (x + 6)^2 = 356$s.
$2x^2 + 12x + 36 = 356$
$x^2 + 6x + 18 = 178$
$x^2 + 6x - 160 = 0$
$x = 10, -16$

The lengths were 10 and 16 yards.

Is it only coincidence that the unusable root in the last two equations was the negative of the second length?

133. A company at a tavern had **L**.8. 15s to pay; but before the bill was paid, two of them sneaked off, when those who remained had each 10s. more to pay. How many were in the company at first?

Let x = original number in company

175/x = number of shillings they would have paid
175/(x-2) = number of shillings they did pay

$\therefore 10 = 175/(x-2) - 175/x$
$10 = 175 (1/(x - 2) - 1/x)$
$10 \cdot x \cdot (x - 2) = 175 \cdot 2$
$x^2 - 2x = 35$
$x^2 - 2x - 35 = 0$
$x = 7, -5$

There were 7 people in the company at first.

134. A grazier bought as many sheep as cost him **L**.60; out of which he reserved 15, and sold the remainder for **L**.54, gaining 2s. a head by them. How many sheep did he buy and what was the price of each?

Let x = number bought

∴ 60/x = price each in pounds

∴ (x - 15)(60/x + 1/10) = 54 What is the 1/10?
(x - 15)(600 + x) = 540x
x^2+ 45x - 9000 = 0
x = 75, -20

∴ 75 sheep purchased at 4/5 pound = 16s. each

135. A and B set out from two towns where were 247 miles apart, and travelled the direct road until they met. A went 9 miles a day; and the number of days, at the end of which they met, was greater by three than the number of miles B went in a day. How many miles did each go?

Let x = number of days travelled

∴ 9x = miles A went
247 - 9x = miles B went
(247 - 9x)/x = miles B went in a day

∴ x - 3 = (247 - 9x)/x
x^2- 3x = 247 - 9x
x^2+ 6x - 247 = 0
x = 13, -19

∴ A went 117 miles and B went 130

136. A person bought two pieces of cloth of different sorts; whereof the finer cost 4s. a yard more than the other; for the finer he paid **L**.18; but the coarser, which exceeded the finer by 2 yards in length, cost only **L**.16. How many yards were in each piece, and what were their prices per yard?

Let x = yards of finer

∴ x + 2 = yards of coarser
∴ 18/x and 16/(x+2) = prices of finer and coarser per yard in pounds

∴ $18/x = 16/(x+2) + 1/5$ What is the 1/5?
$90x + 180 = 80x + x^2 + 2x$
$x^2 - 8x - 180 = 0$
$x = 18, -10$

∴ There were 18 yards of the finer, 20 of the coarser and their costs were **L**.1 and 16s.

137. A set out from C towards D, and travelled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day 1/19th of the whole journey; and after he had travelled as many days as he went miles in one day, he met A. How far apart were C and D?

Let x = distance CD

∴ x/19 = miles B travels per day and number of days before he met A

∴ $x^2/361 + 32 + 7x/19 = x$ What does every term in this line **mean**?

We'll do this the old-school way:

$x^2/361 - 12x/19 = -32$

Complete the square:

$x^2/361 - 12x/19 + 36 = 36 - 32 = 4$

$(x/19 - 6)^2 = 4$
$x/19 - 6 = \pm 2$
∴ $x/19 = 8,4$
∴ $x = 152$ or 76

And both answer the conditions of the problem. So CD = 76 or 152 miles.
We would need more data to determine which one is the actual distance.

138. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 men, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

Let x = number of men in side of hollow square

$\therefore x^2 - (x - 6)^2$ = the whole number of men
$x^2 - (x-6)^2 + 25 = (x^{1/2} + 22)^2$ What does each term here **mean**?
$11x - 44x^{1/2} - 495 = 0$
$x - 4x^{1/2} - 45 = 0$
$\sqrt{x} = 9$ or -5
$x = 81$ or 25

The whole number was 936.

So whats the deal with x = 25?

139. A Mercer bought a number of pieces of two different kinds of silk for **L**.92. 3s. There were as many pieces bought of each kind, and as many shillings paid per yard for them as a piece of that kind contained yards. Now 2 pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let x = number of yards in one piece

$\therefore$ x = number of pieces and also the number of shillings per yard
19 - x = number of yards in the other piece
x^3 and $(19 - x)^3$ = price of each kind

$\therefore x^3 + (19 - x)^3 = 1843$s.
$57x^2 - 1083x + 5016 = 0$
$x = 11, 8$
$19 - x = 8, 11$

Both of which answer the problem. So there were 8 yards in one and 11 in the other and we need more data to know which is which.

140. A square court-yard has a rectangular gravel-walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 92. Required the area of the court.

Start with a diagram and, given this description, expect to get it wrong. All mathematics used to be written out in words like that. Algebra, with its lack of ambiguity, is the reason we got beyond basic mathematics.

Let x = breadth of walk in yards

$\therefore$ 6x - 2 = side of the court
4x - 2 = side of interior square

$\therefore 20x^2 - 8x - 92 = 4(6x - 2)$ What does every term here **mean**?
$20x^2 - 32x - 84 = 0$
$5x^2 - 8x - 21 = 0$
$x = 3, -7/5$

So $(6x - 2)^2 = 16^2 = 256$ is the area of the court.

141. A Merchant bought 54 gallons of Coniac brandy, and a certain quantity of British. For the former he gave half as many shillings per gallon as there were gallons of British, and for the latter 4 shillings per gallon less. He sold the mixture at 10s. per gallon, and lost **L**.28. 16s. by his bargain. Required the price of the Coniac, and the number of gallons of British.

Let 2x = gallons of British

$\therefore$ x = number shillings for one gallon of Coniac
54x = price of all Coniac
x - 4 = number shillings for one gallon British
$2x^2 - 8x$ = price of all the brandy Where does this come from?

$\therefore 2x^2 - 8x + 54x = 10(54 + 2x) + 576$ What does every term here **mean**?
$2x^2 + 26x - 1116 = 0$
$x^2 + 13x - 558 = 0$
$x = 18, -31$

He bought 36 gallons of British, the Coniac cost 18s. per gallon, and the whole price was **L**.48. 12s.

142. During the time that the shadow on a sun-dial, which shews true time, moves from one o'clock to five o'clock, a clock, which is too fast by a certain number of hours and minutes strikes a number of strokes equal to that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during this time. How much is it too fast?

This is one of those puzzles that De Morgan, Todhunter, and other real mathematicians disliked. Do what you can and learn from the method.

Let x = number of hours too fast.

Then the clock strikes (x+2) + (x+3) + (x+4) + (x+5) times in this period
which equals 4x + 14

So the number of minutes = $x^2 + 10 + 25 - 41$ or $x^2 + 10x - 16$

$\therefore x + x^2 + 10x - 16 = 4x + 14$ Explain this model using its **meanings**.
$x^2 + 7x - 30 = 0$
$x = 3, -10$

The number of minutes is 23 so the clock is too fast by 3 hours and 23 minutes.

143. An Upholsterer has two square carpets divided into square yards by the lines of the pattern. He observes, that if he subtracts from the number of squares in the smaller carpet, the number of yards in the side of the other, the square of the remainder will exceed the difference of the number of squares in the smaller carpet, and the number of yards in its side by 88. Also the difference of the lengths of the sides of the carpets is 6 feet. Required the size of each carpet.

Let x = number of yards in side of the lesser

$\therefore x + 2$ = number in side of greater

$(x^2 - x - 2)^2 = x^2 - x + 88$
$(x^2 - x - 2)^2 = x^2 - x - 2 + 90$
$(x^2 - x - 2)^2 - (x^2 - x - 2) - 90 = 0$
$\therefore x^2 - x - 2 = 10, -9$
$\therefore x^2 - x - 12 = 0$ and $x^2 - x + 7 = 0$ Why won't the second one work?
$x = 4, -3$

The carpets contain 16 and 36 yards.

And why does an Upholsterer get his title capitalized when most workers don't? How posh is an upholsterer?

144. A Man playing at hazard won at the first throw, as much money as he had in his pocket; at the second throw, he won 5s. more than the square root of what he then had; at the third throw, he won the square of all he then had; and then he had **L**.112. 16s. What had he at first?

Let x = number of his shillings at first

∴ $2x$ = number he had after first throw
$\sqrt{(2x)} + 5$ = after second throw
$\sqrt{(2x)} + 5 + 2x$ = after third throw
$(\sqrt{(2x)} + 5 + 2x)^2$ = what he won on third throw

∴ $(\sqrt{(2x)} + 5 + 2x)^2 + (\sqrt{(2x)} + 5 + 2x) = 2256$
$(\sqrt{(2x)} + 5 + 2x)^2 + (\sqrt{(2x)} + 5 + 2x) - 2256 = 0$
$\sqrt{(2x)} + 5 + 2x = 48, -47$ and we can ignore the latter

∴ $2x + \sqrt{(2x)} - 48 = 0$
$\sqrt{(2x)} = 6, -7$
$2x = 36, 49$
$x = 18, 49/2$

So he started with 18 shillings.

But what if he started with the 24s. 6d. -- does that work, too?

Redux

145. What is that number, from the square of which, if we take seven times the number, the remainder will be 44?

Just write it out.

Answer: The number is 11. Will the negative root work?

146. Find a number such that if its third part increased by one is multiplied by its half, the product will be 30.

Let x be the number. Then use fractions of x to simply set up the equation.

Answer: 12, -15 Do they both work?

147. The difference of two numbers is 13, and their product exceeds 7 times the less by 135; find the numbers.

As the difference of C of A-B is 13 we can use s and s+13 for the numbers.

Answer: 9, 22

148. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers is equal to 225.

Let x = the first number

Use a : b :: b : c to calculate the second number and then set up the equation.

Answer: 9, 12

149. A draper bought some pieces of cloth for 180 crowns (1 crown = 5s.); had he for the same sum received three pieces more, each piece would have cost 3 crowns less; how many pieces did he buy, and what was the price of a piece?

Let x = number of pieces then 180/x = price of a piece in pounds. Now just set up the equation.

Answer: He bought 12 pieces for 15 crowns each.

150. A gentleman bought a horse for a certain number of pounds; and having sold it for **L**.119, gained as much per cent. as the horse cost him; what was paid for the horse?

Let x = the price of the horse. The "as much per cent. as" is where the x^2 is created.

Answer: The horse cost 70 pounds.

151. Two flocks of sheep, one containing 5 sheep more than the other, were sold for **L**.106. 5s. Each sheep cost as many shillings as there were sheep in the flock; required the number of sheep in each flock.

Let x = smaller flock ∴ x^2 is their cost. Now do the larger flock.

Answer: Flocks had 30 and 35 sheep.

152. A tailor bought a piece of cloth for **L**.73. 10s., from which he cut off 12 yards, and sold the remainder for **L**.64. 15s., gaining 5s. per yard; how many yards were there in the piece, and what did he pay for a yard?

Let x = number of yards in the piece. Convert your two monies to shillings. Cost per yard is money/(something with x in it).

Answer: He bought 49 yards at 30s. per yard.

153. What number is that which being added to its square root, the sum will be 552?

Just don't do anything stupid.

Answer: The number is 529.

154. A grazier bought as many sheep as cost him **L**.50, out of which he reserved 15, and sold the remainder for **L**.54, by which he gained 2s. a head on those sold; how many sheep did he buy, and what did he pay for each?

Let x = number bought. Some of these problems are Pryde's reworkings of Bland's problems. The numbers are different. If you can't get this one, go back and look at how Bland did his. There is no shame in this. Every mathematician learns by relating his problem to a solved problem when possible. This is a trivial instance of that approach.

Answer: He bought 75 sheep and paid 16s. for each.

155. There is a field in the form of a rectangle whose length exceeds its breadth by 20 yards and its area is 6300 square yards; required its length and breadth.

Again, don't do anything stupid.

Answer: 70 and 90 yards which you could read off the area, right?

156. A person being asked his age, answered, if to half my age you add the square root of my age and subtract 12, the remainder is a third of my age; required his age.

Let x = his age. Then let us smack him for being sassy, and don't do anything stupid.

Answer: He is 36 and therefore to old to be such a smart mouth. Smack him again.

157. What number is that, the sum of whose third and fourth parts is less by two than the square of its sixth part?

Let x/3 be its third part and given the "less by two" don't mess up when you set the sides equal.

Answer: The number is 24.

158. The base of a right-angled triangle is less than the perpendicular by 3 and less than the hypotenuse by 6; required the sides.

Let x = the base and use the Pythagorean Theorem.

Answer: Sides are 9, 12, and 15.

159. What two numbers are those whose difference is 3, and whose sum multiplied by the greater equals 405?

You shouldn't need any help here on this by now.

Answer: 12 and 15.

160. There are two square courts paved with stones, each stone a foot square. The side of one court exceeds that of the other by 10 feet, and the two pavements together required 3092 stones; find the lengths of the sides of the courts.

Let the sides be x and x+10.

Answer: 34 and 44.

161. A farmer purchased a number of oxen for **L**.112, and observed that if he had purchased one fewer for the same money, each of them would have cost him 2 pounds more; required the number he purchased and the price of each.

Let x = number purchased and 112/x is then the price he paid for each.

Answer: He bought 8 oxen for 14 pounds a head.

162. A person buys 18 ells of cloth, part blue and part green; for each lot he gave 40s., and he pays for every yard of the blue cloth 1s. per ell more than the green; how many ells of each kind were there?

Let x = ells of blue. Set up the green, the price of both per ell, then the equation. And put the "+ 1" on the correct side.

Answer: 8 ells of blue and 10 of green

163. The length of a rectangle exceeds its breadth by 12, and the sum of the squares of the length and breadth is 20880; what are the sides and area of the rectangle?

Knowing that the area here is the product of adjacent sides, you shouldn't need any help.

Answer: 96, 108, and 10368.

164. Two travellers set out to meet each other from two towns, A and B, which are 120 miles apart; the first goes 6 miles a day, and the other 1 mile a day more than the number of days in which they meet; in how many days will they meet?

Let x = number of days till they meet. Then x+1 is the rate of the second traveller.

Answer: 8 days.

165. Two detachments of foot are ordered to a station 39 miles distant: they begin their march at the same time; but one party, by traveling one-fourth of a mile an hour more than the other, arrives one hour sooner: required the rates of their marches.

Let x = the slower group. And set up the times of each -- to equate -- just like you do prices per yard given the total when the number of yards is a variable.

Answer: Their rates were 3 and 3¼ miles per hour.

166. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first part of 30, and that the sum of the squares of the remaining parts may be 585.

Let x = first part of 30. Then 3x is the first part of 21. Take it from there.

Answer: The parts are 18 and 3, 6 and 24.

167. A regiment of foot was ordered to send 160 men on garrison duty, each company to furnish a like number; but before the detachment marched, three of the companies were sent on another service, when it was found that each remaining company must furnish 12 additional men; required the number of companies in the regiment.

Let x = number of companies. Set up each company's number of men they must supply like a price above. Then you need a price for 3 companies less than that. Then put together your equation.

Answer: 8 companies.

Adfected Ratio and Proportion

168. A Merchant sold a quantity of brandy for **L.**39, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

Let x = price of brandy

∴ 100 : x :: x : solve for the gain = $x^2/100$

We're using the data to get the gain. So 100 parts is to the price as the price is to the (price) × (the price as percent). Now we use the gain with the selling price:

$x^2/100 = 39 - x$

∴ $x^2 + 100x - 3900 = 0$
∴ $x = 30, -130$

∴ 30 = price of brandy of brandy in pounds

I should point out that you don't need a calculator. Go back to

$x^2 = 3900 - 100x$
$x^2 + 100x = 3900$

From $(a + b)^2 = a^2 + 2ab + b^2$ we can view x^2 as a^2 and 100x as 2ab.
This makes b = 50. So we can complete the square as:

$x^2 + 100x + 50^2 = 3900 + 50^2$
$(x + 50)^2 = 6400$
$x + 50 = \pm 80$
$x = 30, -130$

169. Bought two sorts of linen for 6 crowns (1 crown = 5s.). An ell of the finer cost as many shillings as there were ells of the finer. Also 28 ells of the coarser (which was the whole quantity) were at such a price that 8 ells costs as many shillings as 1 ell of the finer. How many ells were there of the finer, and what was the value of each piece?

Let x = ells of finer

∴ x^2 = price of finer in shillings

Where we had as much percent as price it was x·x/100. This is simpler: as many shillings as ells or x·x. Now we want the price of the coarser. We use a proportion of:

ells finer : ells coarser :: cost finer : cost coarser

8 : 28 :: x : solving for this = 7x/2
∴ x^2+ 7x/2 = 30
$2x^2 + 7x = 60$
$2x^2 + 7x - 60 = 0$

∴ x = 4, -15/2

Now the shop probably did not pay him 15 sixpence to take the finer linen.

So x = 4 and the finer costs 16s. and the coarser 14s.

170. Two partners A and B gained **L**.18 by trade. A's money was in trade 12 months, and he received for his principal and gain **L**.26. Also B's money, which was **L**.30, was in trade 16 months. What money did A put in trade?

Let x = money A put in trade

∴ 26 - x = pounds A gained

We use [money both : money A :: gain both : gain A] to get our quadratic.

12x + 16·30 : 12x :: 18 : 26 - x
x + 40 : x :: 18 : 26 - x

∴ 18x = (x+40)(26 - x)
$x^2 + 32x - 1040 = 0$
x = 20, -52

The negative value is senseless.
∴ A put 20 pounds in trade.

171. A and B sold 130 ells of silk, of which 40 ells were A's and 90 B's, for 42 crowns (1 crown = 5s.). Now A sold for a crown one-third of an ell more than B did. How many ells did each sell for a crown?

Let x = number B sold for a crown

∴ $x + \frac{1}{3}$ = number A sold for a crown

$x : 90 :: 1 :$ price of 90 ells $= 90/x$
$x + \frac{1}{3} : 40 :: 1 :$ price of 40 ells $= 120/(3x+1)$

We did those using the first three terms to calculate the fourth.

∴ $42 = 90/x + 120/(3x+1)$
$21x^2 - 58x - 15 = 0$
$x = 3$ or $-5/21$

∴ B sold 3 ells for a crown and A sold $3\frac{1}{3}$ for a crown.

172. Three Merchants, A, B, and C, made a joint stock, by which they gained a sum less than the stock by **L**.80. A's share of the gain was **L**.60; and his contribution to the stock was **L**.17 more than B's. Also B and C together contributed **L**.325. How much did each contribute?

Let x = A's contribution

∴ $x - 17$ = B's contribution
$325 - (x - 17) = 342 - x$ = C's contribution
$325 + x$ = the whole stock
$325 + x - 80 = 245 + x$ = whole gain

∴ $325+x : x :: 245+x : 60$
$245x + x^2 = 60x + 19500$
$x^2 + 185x - 19500 = 0$
$x = 75, -260$

The stocks of A, B, and C were 75, 58, and 267 pounds.

173. The joint stock of 2 partners A and B was **L**.416. A's money was in trade 9 months, and B's 6 months: when they shared stock and gain, A received **L**.228, and B **L**.252. What was each man's stock?

Let x = A's stock

∴ 228 - x = A's gain
416 - x = B's stock
x - 164 = B's gain
∴ 64 = whole gain

9x+6(416 - x) : 9x :: 64 : 228 - x And the four terms **mean** what?
3x+2(416 - x) : 3x :: 64 : 228 - x
192x = (x + 83209228 - x) = 189696 - 604x - x^2
x^2 + 796x - 189696 = 0
x = 192, -988

The stocks were 192 and 224 pounds.

Easier on this one to complete the square...

174. A Vintner sold 7 dozen of sherry and 12 dozen of claret for **L**.50. He sold 3 dozen more of sherry for **L**.10 than he did of claret for **L**.6. Required the price of each.

Let x = price one dozen sherry in pounds

∴ x : 10 :: 1 : number dozens of sherry for 10 pounds = 10/x
10/x - 3 = (10 - 3x)/x = number dozens of claret for 6 pounds

∴ (10 - 3x)/x : 1 :: 6 : price dozen of claret = 6x/(10 - 3x)

∴ 7x + 72x/(10 - 3x) = 50 Find the **meaning** of every term.
292x - 21x^2 = 500
21x^2 - 292x + 500 = 0
x = 2, 250/21

So the price of a dozen of sherry was two pounds and a dozen of claret was three pounds.

Now if you had solved that by completing the square instead of with a calculator, you would have that 250/21 instead of 11.904761904761... which is an approximation no matter how far you extend it. The further you get into math, the more you will appreciate and need the actual and exact fractions. Get a calculator good enough to produce fractions or decimals and set it to fractions.

175. A and B hired a pasture into which A put 4 horses, and B as many as cost him 18s. a week. Afterwards B put in two additional horses, and found that he must pay 20s. a week. At what rate was the pasture hired?

Let x = original number of B's horses

∴ 18/x = pay of each per week in shillings
∴ 72/x = what A paid
72/x + 18 = price of pasture
x + 6 = B's horses in the end

∴ x+6 : x+2 :: 72/x + 18 : 20 :: 72 + 18x : 20x

Do not continue until you know what the last line **means**.

$20x^2 + 120x = 18x^2 + 108x + 144$
$2x^2 + 12x - 144 = 0$
$x^2 + 6x - 72 = 0$
$x = 6, -12$

B had 6 horses to begin with ∴ 72/6 + 18 = 30s. was the price of the pasture.

Redux

176. In a court there are two square grass-plots, a side of one which is 10 yards longer than the side of the other, and their areas are as 25 to 9; what are the lengths of their sides?

Let x = side of the less

$(\text{greater side})^2 : (\text{less side})^2 :: 25 : 9$

Answer: 15, -3 3/4
The negative value can't be any kind of length. But does it work mathematically?

177. A merchant sold a quantity of brand for **L**.39, and gained as much per cent. as the brandy cost him; what was the price of the brandy?

x = cost and gain

Use 100 : x :: x : the gain to calculate the gain. Then set up the equation.

Answer: It cost him **L**.30.

178. The sum of two numbers is 16, and the quotient of the greater divided by the less, is to the quotient of the less divided by the greater as 25 is to 9; find the numbers.

Let x = the lesser. Set up the proportion and then from a:b::c:d the equation is ad = bc.

Answer: The numbers are 6 and 10.

179. Find a number, to the quadruple of which, if 80 be added, the sum will be to the square of the number as 2 to 9.

Let x = the number. Then sum : square :: 2 : 9, right?

Answer: 30

Afterword

I think we can agree that, even when you can do some of them correctly and have some idea of what to do with the ones you can't actually solve, word problems still suck. But it could be worse. I will give two examples. The first is the kind of word problem you would find in the old Cambridge University Tripos, the big hard test for mathematicians.

180. Three horses A, B, C start for a race on a course a mile and a half long. When B has gone half a mile, he is three times as far ahead of A as he is of C. The horses now going at uniform speeds till B is within a quarter of a mile of the winning post, C is at that time as much behind A as A is behind B, but the distance between A and B is only 1/11 of what it was after B had gone the first half mile. C now increases his pace by 1/53 of what it was before, and passes B 176 yards from the winning post, the respective speeds of A and B remaining unaltered. What was the distance between A and C at the end of the race?

Answer: Let 11x = the distance in yards between B and C at the end of the first 1/2 mile, 33x = distance between B and A at the end of the first half mile. When B has gone 1 1/4 miles, B is 3x ahead of A and 6x ahead of C; therefore while B went 3/4 mile or 1320 yards, A went 1320 + 30x yards, and C went 1320 + 5x yards.

Hence, after C increases his pace, the speeds of A, B, C will be proportional to
1320 + 30x, 1320, and 54/53(1320 + 5x) respectively.

Since C passes B when he is 176 yards from the post; therefore while B was going
440 - 176 or 264 yards, C went 264 + 6x.

Therefore	1320 : 54/53(1320 + 5x) :: 264 : 264 + 6x
therefore	1320 + 30x = 54/53(1320 + 5x)
therefore	x(1590 - 270) = 1320
therefore	x = 1

Also it will be found that C's increased pace is equal to A's; therefore there will be the same distance between them at the end of the race as there is when B is 1/4 mile from the winning post, namely 3x or 3 yards.

End of Answer

Oh, yeah. Just shoot me. That was problem number two in Todhunter's chapter for Tripos aspirants. And I chose it because you probably could have figured out the first and easiest one if you had really tried.

The next way in which things could be worse is to have to solve what you can now solve but by doing it without algebra. Not that long ago, educators thought that people your age (assuming you are in middle or high school) were too stupid to understand algebra. So they kept you in the tenth century where there was no algebra and you had to solve the same problems in a much worse way. Let me show you.

181. A man lost 3/5 of his sheep; now he finds 5 and sells 3/5 of what he has for cost price, and receives $18; but if he had lost 5 and sold 3/5 of the remainder for cost price, he would have received $6; how many sheep had he at first?

Answer:

5/5 = number of sheep he had at first.
3/5 = the number he lost.
5/5 - 3/5 = 2/5, the number he had after losing 3/5.
2/5 + 5 = number he had after finding 5
3/5 of (2/5 + 5) = 6/25 + 3 = the number he sold
2/5 - 5 = the number had he lost 5
3/5 of (2/5 - 5) = 6/25 - 3 = number he would have sold
$18 = what (6/25 + 3) sheep cost
$6 = what (6/25 - 3) sheep cost
$12 = $18 - $6 = what (6/25 + 3) sheep minus (6/25 - 3) sheep or 6 sheep cost
$2 = 1/6 of $12 = what one sheep cost
$18 = what 18/2 or 9 sheep cost
But $18 = what (6/25 + 3) sheep cost.
Therefore 6/25 + 3 sheep = 9 sheep
Therefore 6/25 = 6 sheep
Therefore 1/25 = 1/6 of 6 sheep = 1 sheep
Therefore 25/25 = 25 times one sheep = 25 sheep = number he started with

Oh, yeah. That was easy to figure out. Next time you hear some academic bozo saying students are too stupid to do algebra, you can send him back to the tenth century where he can discover what other progress has been made in the interim. The academic recently in the news for saying such a thing thinks that only the mathematically-talented can understand quadratics. Those are too hard for ordinary people. That makes all of us here geniuses, right? Or him really stupid. You decide.